1-1-1994

Atomism and infinite divisibility.

Ralph E. Kenyon
University of Massachusetts Amherst

ATOMISM AND INFINITE DIVISIBILITY

A Dissertation Presented

by

RALPH E. KENYON, JR.

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

February 1994

Philosophy Department

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RALPH E. KENYON, JR.

Approved as to style and content by:

Bruce Aune, Chair

Gareth Matthews, Member

Gary Hardegree, Member

Emmon Bach, Member

John Robison, Department Head
Philosophy

ACKNOWLEDGEMENTS

I wish to acknowledge the mentorship of Bruce Aune, without which this project could never have been completed. While we have differences, I have felt a kinship to his brand of philosophy. Bruce's many hours of patient discussion have guided me to where I am now. I wish also to thank Gareth Matthews, Gary Hardegree, and Emmon Bach for their assistance and patience in the development of this work.

I am also indebted to A. E. Van Vogt, whose novel *The World of Null-A* first introduced me to general semantics and Alfred Korzybski. It was through the medium of Korzybskian general semantics and its emphasis on Karl Popper that I became intensely interested in the philosophy of science and philosophy in general.

I wish also to acknowledge the influence of the writings of Russell Ackoff, Nina Bull, Patricia Churchland, Fred Dretske, Fred Emery, Douglas Hoffstadter, Thomas Kuhn, Jean Piaget, Karl Popper, and Karl Pribram, all of whom have contributed significantly to my present view.

Finally, I wish to thank Virginia Sturtevant for her effort in editing this document and her patience with me through the development of this dissertation.

ABSTRACT

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FEBRUARY 1994

RALPH E. KENYON JR., B.A., B.S., MIAMI UNIVERSITY,

M.A., PEPPERDINE UNIVERSITY, M.S., OLD DOMINION UNIVERSITY

M.A, UNIVERSITY OF MASSACHUSETTS AMHERST

Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Bruce Aune

This work analyzes two perspectives, Atomism and Infinite Divisibility, in the light of modern mathematical knowledge and recent developments in computer graphics. A developmental perspective is taken which relates ideas leading to atomism and infinite divisibility. A detailed analysis of and a new resolution for Zeno's paradoxes are presented. Aristotle's arguments are analyzed. The arguments of some other philosophers are also presented and discussed. All arguments purporting to prove one position over the other are shown to be faulty, mostly by question begging. Included is a sketch of the consistency of infinite divisibility and a development of the atomic perspective modeled on computer graphics screen displays. The Pythagorean theorem is shown to depend upon the assumption of infinite divisibility. The work concludes that Atomism and infinite divisibility are independently consistent, though mutually incompatible, not unlike the wave/particle distinction in physics.

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PREFACE

The belief that there are indivisible units of extension is termed 'atomism'. The opposing view is that extension is infinitely divisible. For the purpose of this work, I shall occasionally refer to this position as 'divisionism' and the adherents to this view as 'divisionists'.

Preliminary studies showed that some of the traditional arguments supporting infinite divisibility make use of premisses which effectively beg the question. The same appears true of Atomism. In this work I show that the traditional mathematical arguments for infinite divisibility are flawed and that most philosophers in the past did not discover the flaw. My view is that the two positions, atomism and divisionism, are each internally consistent and, though mutually incompatible, are independent in a way not unlike Euclidean and non-Euclidean geometries or the waves and particles of quantum physics.

CHAPTER I

PRELIMINARY NOTIONS

Introduction

People have been dividing things and sharing them for millennia. When it comes to dividing something among two or three persons, the "right amount" can often be cut or broken off at once. But when one is dividing something among four persons, it is often easier to divide it in half and then divide each half in half again. Division into smaller portions is most often achieved by a process of repeated or successive division. At some time long past it must have occurred to someone to wonder how long such a process could be continued.

Practical experience sometimes suggests that there is a limit to the process. In dividing something, and dividing the results again, sooner or later one reaches a limit where the remaining parts cannot practically be divided again. For some things this limit is much more obvious than others. Dividing a bag of marbles among children provides an example of an obvious limit. But practical experience also suggests that sometimes there are non-obvious cases. Dividing a pitcher of liquid refreshment among imbibers provides an example of this. One might resort to counting drops, but drops come in different sizes, and there is the matter of the residual film of liquid. We can easily conjecture that the liquid could continue to be divided beyond our ability to distinguish the divisions.

We can use a magnifying glass and a razor blade to divide a droplet of water that we could not perceive as large enough to divide when we looked at it with only the naked eye. We can also use a microscope and appropriately sized tools to divide the droplet that seemed too small to divide when we used only the magnifying glass. The use of higher and higher powers of magnification shows that at each stage an apparently indivisible droplet proved to be divisible when it was looked at with greater magnification. While very high powered devices have been able to distinguish the individual atoms of heavy metals, no such results have been obtained with water. It is our atomic *theory* of matter that allows us to conclude that there is also an indivisible minimum size for water.

Apart from that modern atomic theory, we can easily generalize that the process of successive division need have no end -- that the process can continue indefinitely. But we are quite aware that our perception is limited. There are smallest amounts -- less than which we cannot perceive.

Humans, being the divisive people we are, take sides and argue about such questions. We may reasonably infer that ancient peoples were divided in their opinions even before recorded history.

Atomism is the view that successive division must terminate in some indivisible minimum. The opposing view is the belief that successive division can be continued infinitely. Since division is a process that is applied to something, an immediate dichotomy concerning the question is possible. The question may be asked with emphasis on the object of the process, or it may be asked with emphasis on the process itself. It is from the former that the name 'infinite divisibility' derives and is given to the view opposing atomism. I shall sometimes refer to that view as 'divisionism'. Divisionism is most often expressed as the view or belief that matter or extension is infinitely divisible. What may be the proper object of the process has varied with the major philosophical positions.

I have already hinted that recognition or perception might influence knowledge of divisibility. I will touch on the epistemological concerns relating to the arguments, but I will primarily be focusing on the metaphysical aspect of the question. The Epicureans argued from the perceptible to the imperceptible by analogy. More recently the question was applied to the conceivable.

One way to organize the perspectives taken by concerns for metaphysical questions, epistemological questions, and questions regarding conceivability is along a subjective-objective dimension. Philosophical perspectives fall along that dimension with realism toward the objective end, idealism toward the subjective end, and phenomenalism somewhere between these two.

In the context of realism, one asks whether matter, extension (space), and duration (time) are infinitely divisible. In the context of phenomenalism, one asks whether perceptions are infinitely divisible. In the context of idealism, one asks whether concepts are infinitely divisible. If one is to focus on the process itself, questions concerning the meaning of 'infinite' arise.

All these questions can be asked with a decidedly metaphysical flavor as well as with a decidedly epistemological flavor, but realism lends itself much more easily to metaphysical questions while phenomenalism lends itself much more easily to epistemological questions. And idealism lends itself more easily to questions regarding conceivability. The analogical relationships I see suggest the visual representation illustrated in figure 1.

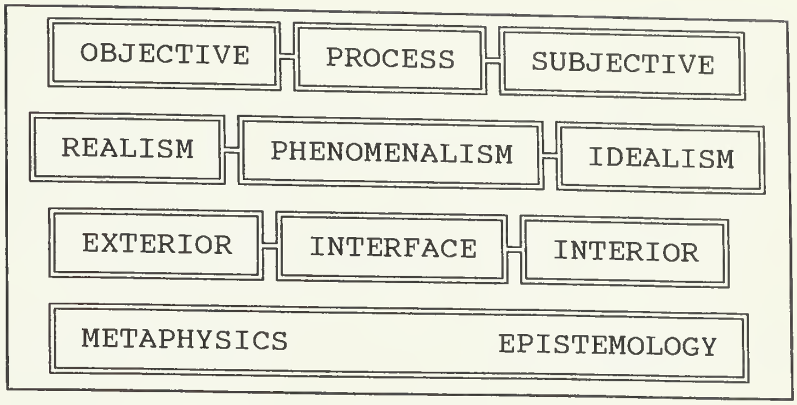

Figure 1. Subjective-Objective dimension.

In this exposition I shall be concerned mostly with the validity of various arguments for and against each position. I am particularly concerned with mathematical arguments that have been presented and the light they shed on premisses which have been used to support one or another position. To "cut to the chase", my research suggests that there is no valid argument with true premisses which establishes one position over the other. It seems that there are consistent models for both positions. And these models differ by one "axiom" -- the presumption of atomism on the one hand or of infinite divisibility on the other hand.

I also presume a "developmental" perspective consistent with genetic epistemology.[1] I assume that, for the most part, earlier writers assimilated or understood by means of fewer distinctions, and that some problems with earlier views may be resolved by more recent distinctions. However, there are instances when the mere addition of a distinction is insufficient to resolve the issues. The mappings of concepts may have to be significantly reorganized in order to accommodate a newer development.[2] This developmental perspective is also suggested by Furley when he traces the evolution of Atomism as presented by Epicurus:

> [This] essay will show how Epicurus' doctrine evolved; it is a modification, adopted for the purpose of meeting Aristotle's criticisms, of a doctrine which the earlier atomists put together to meet and thwart the Eleatic attack on pluralism.[3]

Some of the earliest writings on this subject are attributed to Zeno of Elea. By Zeno's time the controversy was fairly well developed; the positions were characterized as beliefs in "atomism" and "infinite divisibility". These contrasting beliefs are closely related to the earlier question, whether "things are one" or "things are many". If things are infinitely divisible, any division into parts yields parts which are themselves divisible into parts -- "All things are many" (all the way down). If things are not so divisible then there are things that are not many -- "things are one" (and indivisible).

Even earlier than Zeno, Heraclitus had things to say about the controversy. Interestingly enough, Heraclitus seems to have had the most mature views on the topic, although records of his thoughts are the most scanty. (See page 17.)

Mathematical Induction

Mathematical induction figures prominently in my analysis of infinite divisibility, and I would be remiss not to briefly present it here. Mathematical induction is applied to a statement which is expressed in terms of some arbitrary natural number, usually represented by 'N'. Induction, as one might expect, is a way of reasoning to conclusions about more particulars than may reasonably be examined. Mathematical induction has two premisses and one conclusion. One premiss that must be satisfied is that the statement in question be true for some small value of N. This need not be the smallest, but it usually is and most often is the number 1. The second premiss that must be satisfied is a material conditional going from an arbitrary number N (equal to or larger than the value used in the first premiss) to the next larger number N+1. If these two premisses are satisfied, then one may draw the conclusion that the statement is true for all values of N (greater than or equal to the small value of N). Suppose we refer to the statement as S(X). Premiss 1 would be expressed:

S(A) is true.

Premiss 2 would be expressed:

IF S(N) is true THEN S(N+1) is true (whenever $N \geq A$).

The conclusion that may be drawn is:

For ALL X ($\geq A$), S(X) is true.

This conclusion is justified by the following reasoning. Suppose A is 1. S(2) may be inferred from S(1) and premiss 2 by modus ponens. S(3) may be inferred from S(2) and premiss 2 by modus ponens. This process may be continued until X is as large as you like. Mathematical induction is the shortcut method for deducing the truth of S(X) for all X. Simply by showing that premisses 1 and 2 are satisfied for some statement, S(X), we may use mathematical induction to directly demonstrate the truth of S(K) without going through K-1 applications of modus ponens. Because mathematical induction is only a shorthand for deductive applications of modus ponens, it is strictly truth preserving, as are all valid deductive arguments.

Infinity

In the literature three primary senses are given for the term "infinite" and its derivatives. These are *arbitrarily large*, *unending*, and *aleph null* ($\aleph_0$). Distinguishing carefully among these senses takes one a long way toward resolving Zeno's paradoxes. In many instances substituting the appropriate phraseology in a premiss statement using the term "infinite" renders the truth value of the premiss much more apparent.

The first sense, *arbitrarily large*, is illustrated by James Thomson in "Tasks and Super-Tasks".

> [T]o say that a lump is infinitely divisible is just to say that it can be cut into any number of parts.[4]

Infinity is paired with *any number* with the implicit understanding that this number may be as large as you like.

The second sense, *unending*, is illustrated by Russell.

> Etymologically, 'infinite' should mean 'having no end'.[5]

The third sense, *aleph null*, is most precisely captured by the axiom of infinity in the Zermelo-Frankel set theoretic representation for numbers[6]. That axiom is an existence axiom in that it postulates the existence of a number with certain properties. The axiom of infinity can be paraphrased in terms of ordinary natural numbers as follows: There is a number X (infinity) such that 1<X and whenever N<X then N+1<X, where N is any natural number generated from 1 by repeated applications of the successor axiom (+1).

This axiomatic definition for infinity ($\aleph_0$) explicitly utilizes the structure of mathematical induction.

Motion

The notion of "motion" or "velocity" figures into atomism versus divisionism arguments in a number of ways. It is implicit in two of Zeno's arguments and explicit in a third. It is appropriate to present a brief view of the current space-time perspective on motion to provide an explicit background for understanding the arguments presented as they reflect on it.

In mathematical physics *velocity* ("motion") is defined as the rate of change of position with respect to time. When time is taken as a fourth dimension, and one looks at events as having both spatial and temporal coordinates, no "motion" can be seen. When one adopts such a "four-dimensional space-time perspective", one attends to a three-dimensional "object in motion" as a four-dimensional space-time "worm" with its "starting position" at one place-time and its "ending position" at another place-time. The starting position is identified by its having a "lower" time-coordinate. From the four-dimensional space-time perspective a "velocity" is seen as just the slope of a line drawn with both space and time coordinates. It is no different from the rate of change of one spatial dimension with respect to another, such as the slope of a road. For example, on a road with a 7% grade, the road rises 7 feet for every 100 feet of length. Saying the road "rises" is only valid in respect to one's position along the road (and whether one is going up the road or down the road). In order to develop the analogy with motion, references to time must be removed from the notion of physical slope. As one stands in different positions along the road, one's elevation varies depending on one's position.

The physical slope of the road corresponds directly to the ratio of position with respect to time. When an object in three-space is moving in time its spatial coordinates are changing, but only as the time coordinates are changing. Its position coordinate varies with its time coordinate just as one's elevation coordinate varies with one's position coordinate along the road.

When one looks at an object from the four-dimensional space-time perspective one sees all the space-time coordinates of the object. Both the starting point and the ending point coordinates are immediately available. The view is one that could be called "omniscient" in that all space-time positions can be seen "at once".[7] The analogous perspective for viewing the road is from the side. By standing to one side of the road with a 7% grade, far enough back, one can see both the bottom and the top of the hill "at once". In a like manner, "standing to one side of time" allows one to adopt the four-dimensional space-time perspective and see both the beginning and end of the motion of an object "at once".

From the four-dimensional space-time perspective no "motion" is seen at all -- thus exonerating the ancient argument that motion is impossible. However I will discuss the questions mostly from the more conventional, three-dimensional perspective. Keeping in mind the way motion is defined in mathematical physics will provide a consistent view of the problems of infinite divisibility and atomism.

Developments Leading Toward Atomism

The concept of the atom did not emerge on the scene full-blown. It evolved from a number of earlier views through a gradual process involving a number of stages. Atoms, as we know them, and as most clearly presented by Lucretius, are conceived of as hard, indivisible bits of solid matter that come in various shapes and kinds. They whiz around in empty space colliding with each other, sometimes bouncing off and sometimes sticking to each other.

All the "stuff" of the universe is made up of them. Atoms could not exist as even a concept were it not for the co-existence of empty space into which to put them. The universe can be seen as distinguished into bits of matter and space. But without the notion of empty space, the notion of atoms cannot exist.

A "solid" concept of atomism also requires some stability concerning the questions what the stuff of existence may be made of and in how many kinds it can come. I will touch only briefly on this question as it is peripheral to my main interest. But developments in this argument do affect the development of atomism proper, so I present a brief summary.

Atoms (of matter) also represent a synthesis of the notions of dividing and not dividing. The stuff of the universe is divided into bits (atoms), but the atoms themselves can not be divided. Required also is some notion of "size" for matter. Arriving at a birth for the concept of atomism requires that all these questions have undergone some development and some resolution. And all these developments depend upon some notion of existence.

I will not be touching deeply on these early developments. But I will outline them briefly to establish a context for the main discussions. My main focus is on the mathematical arguments that arise as a result of these early arguments.

"Being" or Existence

The question of "being" permeated the early pre-Socratic philosophy. The convoluted arguments centered around what appeared then to be worse than an oxymoron -- the apparently contradictory act of asserting the existence of something in order to deny it. The act of speaking or even thinking something was viewed at the time to have had existential import.

> when the goddess points out to her listener that he could neither know nor point out what-is-not (2.7-8), she is precluding reference in thought or speech to the non-existent.[8]

This made talk of "nothing" or non-existence very problematic. It was the denial of this "void" that lead to monism. In the denial of nothing the early Ionians concluded that everything was one and that motion was impossible. Atomism has its roots in this concept of "the one" or unity -- which later became associated with the idea of indivisibility.

Thales and Anaximander

Thales of Miletos is credited with having explained that everything is made of water; that air, ether, and even earth are just different forms of the one substance. As a result, Milesian thought was dominated by corporeal monism[9], that all things reduce to the one (body) which appears in different forms.[10]

> All the Ionians had taken for granted that the primary substance could assume different forms, such as earth, water, and fire, a view suggested by the observed phenomena of freezing, evaporation, and the like. Anaximenes had further explained these transformations as due to rarefaction and condensation (§ 9).[11]

Thales is credited with opening the question that leads to the atomic theory.[12] It might be reasonable to attribute the contrasting view to Anaximander, of the next generation of Milesians, who was a follower of Thales.

> Thales, Anaximander seems to have argued, made the wet too important at the expense of the dry.[13]

Burnet credits Anaximander with giving some equal standing to the different "elements".

> [It] is more natural to speak of the opposites as being 'separated out' from a mass which is as yet undifferentiated . . . .[14]

But he also begs the question by suggesting that this somehow entails it being made of "particles".

> That, of course, really implies that the structure of the primary substance is corpuscular, and that there are interstices of some kind between its

> particles. It is improbable that Anaximenes realised this consequence of his doctrine.[15]

No such conclusion is warranted without some presumption of the incompressibility of matter, a later atomic development. The mixing (and separation) of colors shows a non-particulate counter-example. Burnet seems to have "projected" a more modern view into his analysis.

Already the Ionians have a general question regarding whether there is some primary stuff of existence that is divisible into other substances, or there is one of these that cannot be so divided. Anaximander affirmed divisibility while his predecessor Thales affirmed the one.

Pythagoras

The Pythagoreans taught that all things were number. And number is an expression of unity or oneness -- the Milesian monism in a less corporeal form. Pythagoras, who was in Kroton from about 532 B.C. to the end of the sixth century, was probably a disciple of Anaximander.[16] He is credited with discovering the problem of doubling the square.[17]

> "Pythagoras discovered that the square of the hypotenuse was equal to the squares on the other two sides; but we know that he did not prove this in the same way as Euclid did later (I.47). It is probable that his proof was arithmetical rather than geometrical; and, as he was acquainted with the 3 : 4 : 5 triangle, which is always a right-angled triangle, he may have started

> from the fact that $3^2 + 4^2 = 5^2$. He must, however, have discovered also that this proof broke down in the case of the most perfect triangle of all, the isosceles right-angled triangle, seeing that the relation between its hypotenuse and its sides cannot be expressed by any numerical ratio. The side of the square is incommensurable with the diagonal."[18]

In the atmosphere of Milesian monism, it must have been quite disconcerting to be unable to find whole numbers giving a ratio for doubling the square. With monism firmly established in the culture, the faith that such a number existed and would be found probably prevented the discovery of what is now known to be $\sqrt{2}$. Had it been discovered then, atomism might have been dealt a disabling blow. If the square root of two is to be a number, then number can no longer be strictly a unity.

Heraclitus and Parmenides

Heraclitus of Ephesus (fifth century) was known for his theory of flux and his doctrine of the unity of opposites. Most of Heraclitus's works are lost, but a few fragments have been gleaned from various sources; he was quoted by ancient philosophers from Plato on. Enough substance is contained in those fragments to provide a reasonable assessment of his view concerning the present question. The ancient question concerned whether all things were "one" or "many". We may understand "many" to mean "composed of parts" where the term 'parts' is used circularly or recursively. Zeno's *paradox of plurality* (page 41) explicates this issue more fully. By "one" we may understand "an indivisible whole". While this pairing may not be exact, I think it naturally evolved into the atomism versus infinite divisibility distinction. Atomism may be an early attempt to resolve the paradox of plurality. If so, it would provide for a true recursive definition for the term 'part'. A *part* is either an atom or it is something composed of smaller parts. [See the discussion of the Paradox of Plurality on page 41 below.]

For Heraclitus, the question whether all things are one or many is answered by fragment 112:

> From out of all the many particulars comes oneness, and out of oneness comes all the many particulars.[19]

While this hints at Hegel's synthesis of thesis and antithesis, it also suggests that anything that is one is also divisible. One could interpret this as an affirmation of infinite divisibility. But his doctrine of the unity of opposites actually mandates that he affirm both views. His theory of flux, in which things are continually changing into their opposites, is moderated by his principle of balance. That principle is best stated in fragment 33 and can be understood as a conservation law.

> [The] resultant amount is the same as there had been before.[20]

On this account, it would seem, Heraclitus would not have subscribed to the naive view that if something were infinitely divisible then it would be divisible into nothing at all. Such a premiss would violate his principle of balance. Yet that premiss is exactly the one which has gone unchallenged for millennia. I shall return to this premiss when it is more explicitly stated. [See the discussion under the paradox of plurality on page 73 below.]

Parmenides taught, in opposition to Heraclitus, that being was "a solid, homogeneous, extended body",[21] and that it was spherical and unchanging. In many ways this reaffirms the earlier Milesian view of the one, but the similarity to the later conception of an atom is readily apparent.

Empedocles and Anaxagoras

Empedocles seems to have attempted to synthesize the views of his predecessors. Thales made everything out of water; Heraclitus made everything out of fire; Anaximenes made everything out of air[22]; Anaximander gave none of these primacy. Empedocles made everything out of all of these, including earth, and, as such, provided the first theoretical forerunner of modern atomic chemistry.

> Empedocles called his elements 'roots', and Anaxagoras called his 'seeds', but they both meant something eternal and irreducible to anything else, and they both held the things we perceive with the senses to be temporary combinations of these.[23]

But Anaxagoras apparently thought Empedocles's system flawed. He did not think that four elements could produce all the substances we see. According to Burnet, Anaxagoras was misunderstood by both Aristotle and the Epicureans. Burnet explains that Anaxagoras's "seeds" were infinitely divisible but differed in their proportions.

> He therefore substituted for the primary 'air' a state of the world in which 'all things (χρήματα) were together, infinite both in quantity and in smallness' (Fr. I). This is explained to mean that the original mass was infinitely divisible, but that, however far division was carried, every part of it would contain all 'things' (χρήματα), and would in that respect be just like the whole.[24]

The "flavor" of the disputes of the time were perhaps eloquently expressed by Zeno in his paradoxes, to which we now turn.

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3. David J. Furley, Two Studies in the Greek Atomists, (Princeton: Princeton University Press, 1967, p. 21.

4. James Thomson, "Tasks and Super-Tasks", in Zeno's Paradoxes, ed. Wesley Salmon, (Indianapolis: Bobbs-Merrill, 1970), p. 91.

5. Bertrand Russell, "The Problem of Infinity Considered Historically", in Zeno's Paradoxes, ed. Wesley Salmon, (Indianapolis: Bobbs-Merrill, 1970), p. 55.

6. Keith J. Devlin, Fundamentals of Contemporary Set Theory, (New York: Springer-Verlag, 1979), p. 52.

7. This risks confusing the issue by introducing a fifth dimension (of time). "At once" implies "at one time", but the four-dimensional space-time perspective is not limited to "one time". In order for "at once" to be meaningful in this new context, it must mean at one time along some temporal dimension. Since we are already looking at all the time coordinates, that one time must be along another dimension of time, a fifth dimension. Since we are not accustomed to adopting the four-dimensional space-time perspective, we must be especially on guard to prevent the intrusion of language not consistent with that perspective.

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18. Burnet, p. 43.

19. Philip Wheelwright, Heraclitus, (Princeton: Princeton University Press, 1959; reprint, Athenaeum edition, 1971), p. 90.

20. Wheelwright, p. 37.

21. Clark, p. 28.

22. Burnet, p. 19.

23. Burnet, p. 55.

24. Burnet, p. 62.

CHAPTER II

ZENO'S PARADOXES

Arguments about atomism and infinite divisibility were first developed in detail by Zeno of Elea (born c. 490 bc) in the form of his now famous paradoxes. Since these paradoxes have had a very important influence on subsequent disputes regarding atomism and divisionism, it is important to identify them here. I shall give the basic thrust of each paradox by a short statement, and then, in order to show the logical form of the underlying argument, I shall offer an expanded version of it. I shall defer my critical comments on the paradoxes and the relevance to my subject to the next chapter.

The Achilles

I can succinctly state the Achilles paradox as follows: Achilles and the tortoise race; the tortoise is given a head start. By the time Achilles reaches the spot where the tortoise was, the tortoise will have moved on. So Achilles can never catch the tortoise.[1]

To appreciate the logical structure of the paradox, the relevant premisses and conclusions must be identified and set forth in explicit terms. Some of the premisses are merely implicit in the brief statement of the paradox.

A basic assumption of the argument is this:

AP0* To catch the tortoise, Achilles must eventually occupy the same spot as the tortoise.

This premiss is present in the argument in a contrapositive form:

AP0′ If it is always the case that Achilles does not occupy the same spot as the tortoise, then Achilles never catches the tortoise.

The information given in the argument ostensibly does not permit the conclusion that Achilles does catch the tortoise. What the premisses do ostensibly warrant is indicated as follows:

The initial premiss is:

AP1 The tortoise is given a head start.

---

* Premiss, inference, and conclusion statements will be labeled according to the following scheme: The first letter or letter and digit will identify the argument; the second letter will identify the statement type; and the third digit will be a sequence number. A sub-sequence number may also be used in some cases.

First letter or letter and digit		Second letter	
The Achilles	A	Premiss	P
The Dichotomy form 1	D1	Inference	I
The Dichotomy form 2	D2	Conclusion	C
The Arrow	R	Definition	D
The Stadium	S		
Paradox of Plurality	P		

AI1 AP1 ==> AC1

AC1 The tortoise is ahead of Achilles.

AP2 If the tortoise is ahead of Achilles then Achilles runs toward the tortoise.

AI2 AC1 & AP2 ==> AC2

AC2 Achilles runs toward the tortoise.

AP3 If Achilles runs toward the tortoise, then the tortoise runs on ahead to another spot.

AI3 AC2 & AP3 ==> AC3

AC3 The tortoise runs on ahead to another spot.

AP4 If the tortoise runs on ahead to another spot then, when Achilles reaches the spot previously occupied by the tortoise, the tortoise will occupy a spot different from the one Achilles occupies.

AI4 AC3 & AP4 ==> AC4

AC4 The tortoise occupies a spot different from the one Achilles occupies.

AP5 If the tortoise occupies a spot different from the one Achilles occupies, then the tortoise is ahead of Achilles.

AI5 AC4 & AP5 ==> AC5

AC5 The tortoise is ahead of Achilles.

But AC5 is just AC1; the argument leads to the original premiss. No valid reasoning with these premisses leads to any other conclusion. No matter how many times the argument is followed through, the conclusion is always the same -- the tortoise is ahead of Achilles. No premiss or conclusion in the argument leads to Achilles being in the same spot as the tortoise.

We could argue that what is true of each instant is true of the whole race, but that would involve the fallacy of composition. <u>A stronger way to conclude</u> that Achilles can never catch the tortoise is to use mathematical induction on each iteration of the argument. AC1 is true after the first iteration. Assume AC1 is true after N iterations. Then, by applying the premisses in order, AC1 is also true after N + 1 iterations. These two conditions satisfy the requirements for mathematical induction and allow us to conclude that it is always the case (after every iteration) that Achilles does not occupy the same spot as the tortoise. (If always AC4, then Achilles can never catch the tortoise.) However, since mathematical induction was not available to Zeno, we may appeal to additional premisses.

AP6 There is no end to an infinite sequence (of steps or acts).

AP7 Achilles's repeated attempts to catch the tortoise constitute an infinite sequence.

AI6 AP6 & AP7 ==> AC6.

AC6 There is no end to Achilles repeated attempts to catch the tortoise. In other words, Achilles can never come to the end of his sequence of attempts to catch the tortoise -- that is, Achilles can't catch the tortoise.

It is interesting to note that this argument is valid without reference to the speeds of either Achilles or the tortoise. It is clear that if Achilles runs slower than the tortoise or at the same speed as the tortoise then we have no difficulty with the conclusion. But if Achilles runs faster than the tortoise the conclusion is absurd. Since Achilles is "the fleetest of all Greek warriors"[2], we may assume:

AP8 Achilles runs faster than the tortoise.

AP9 If Achilles runs faster than the Tortoise, and he runs toward the tortoise, then Achilles will be closer to the Tortoise when he reaches the spot the tortoise left.

With these two additional premisses, it is possible to conclude, validly, that Achilles is always getting closer to the tortoise.

AI7 AC2 & AP8 & AP9 => AC7

AC7 Achilles is closer to the tortoise.

But nothing in these premisses allows us to conclude that Achilles actually catches the tortoise. The paradox is that an apparently valid argument with acceptable premisses yields such an unacceptable outcome.

The Dichotomy

The dichotomy has two forms.

1. For Achilles to reach any point he must get half way. Then he has to get half the rest of the way. Since there will always be a fraction to go, he can never reach any point.

This argument must also be unpacked and stated in the form of premisses and conclusions. Here are the relevant premisses:

D1P1 For Achilles to reach another point (his destination) he must first get half way to it.

D1P2 If Achilles is at a point not his destination, then he moves toward his destination point.

D1P3 If Achilles moves toward his destination point then he first reaches a point half way towards it.

D1P4 If Achilles is at a point half way towards his destination, then he is at a point which is not his destination.

D1P5 Achilles is at a point which is not his destination.

And here is the form of the argument:

D1I1 D1P2 & D1P5 => D1C1

D1C1 Achilles moves toward his destination.

D1I2 D1C1 & D1P3 => D1C2

D1C2 Achilles reaches a point half way toward his destination.

D1I3 D1C2 & D1P4 => D1C3

D1C3 Achilles is at a point which is not his destination.

But D1C3 is just D1P5. As in the Achilles, the argument leads to the original premiss. No valid reasoning leads to any other conclusion. No matter how many times the argument is followed through, the conclusion is always the same. Nothing in the argument leads to Achilles being at the other point. As in the Achilles, this form of the Dichotomy rests on the premisses (1), that an infinite series has no end and (2), that Achilles is traversing an infinite series of points. Since an infinite series has no final term it is concluded that Achilles cannot reach the end of the series. But this is just what he must do in order to reach the other point.

2. The second form of the paradox can be explained this way:

For Achilles to reach any point he must get half way. To get half way, he must get to half that, etc. To get anywhere, he must have already covered an infinite number of points.[3]

In this form of the Dichotomy, the infinite series of points that Achilles is to traverse is "reversed" from that of the first form. As the points are enumerated, the second is one quarter of the way to the destination in the second form while it is three quarters of the way to the destination in the first form. Between Achilles and any other point there is an infinite series of points. For Achilles to get to any one of these points, he must have already traversed, in reverse order, the infinite series of points which, according to the premiss in the first form, he can not come to the end of. This form of the Dichotomy entails a premiss which is less clearly acceptable.

D2P1 Achilles cannot traverse an infinite number of points.

D2P2 Between Achilles and any point is an infinite number of points.

D2P3 To traverse the distance to any point, Achilles must traverse all intervening points.

D2I1 D2P2 & D2P3 => D2C1

D2C1 To traverse the distance to any point, Achilles must traverse an infinite number of points.

D2I2 D2C1 & D2P1 => D2C2 (Modus Tolens)

D2C2 Achilles cannot traverse the distance to any point.

This argument is more straight forward, but it is also clear that premiss D2P1 is not obviously true. However, this argument and the preceding first form of the Dichotomy exhibit symmetry. Support for premiss D2P1 can be achieved with a additional premisses -- namely:

D2P4 Achilles can traverse an infinite sequence of points if and only if he can come to the end of a infinite series.

D2P5 Achilles can not come to the end to an infinite series.

D2I3 D2P4 & D2P5 => D2P1 (Modus tolens)

D2P1 falls out as a conclusion from these two less questionable premisses. Since it is clear that Achilles can not come to the end of an infinite sequence, it must also be the case that he can not traverse an infinite sequence of points.

The first form of the Dichotomy concludes that Achilles can't reach any point, and the second form concludes that he can't even get started. Although the arguments appear to be valid, both conclusions are clearly absurd.

The Arrow

The arrow cannot move. To do so requires that it be in one place equal to itself during one part of an instant and another during another. Also, it would occupy a space larger than itself in order for it to have room to move.[4,5]

RP1 Everything at a place equal to itself is at rest.

RP2 A flying arrow is always at a place equal to itself at every instant in its flight.

RI1 RP1 & RP2 => RC1

RC1 A flying arrow is at rest at every instant in its flight.

RP3 That which is at rest at every instant does not move.

RI2 RC1 & RP3 => RC2

RC2 A flying arrow does not move.

While the above rendition of the argument suffers from the fallacy of composition, it is possible to render the argument in a form not subject to this fallacy. This can be done as follows:

Def: RD1 An *instant* is an indivisible minimal element of time.

Def: RD2 Something is *at rest* (instantaneously) iff it is in its place (one place equal to itself) in one instant and it is in the same place (equal to itself) in different instants (remains at rest).

Def: RD3 Something *moves* iff it is not at rest.

RI3 RD2 & RD3 => RC2

RC2 Something moves iff either (A) it is not in one place (equal to itself) in one instant or (B) it is in different places (equal to itself) in different instants.

I will present the two disjuncts as separate cases.

Case 1:

(A) That which moves is not in one place (equal to itself) in one instant.

It would appear that this case could be disposed of immediately by noting that it seems to contradict RP2 directly. -- RP2 could be interpreted that everything is always in a place equal to itself. -- However, it is instructive to analyze more deeply. We can consider the 'not' as applying to "one place" or alternatively to "one instant". Let us first consider the 'not' applied to "one place". "Not one place" becomes "different places".

RI42 RC2 => RC22

RC22 Something moves iff it is at different places in the same instant. (An instant has more than one place.)

Although this interpretation is practically inconceivable to us, it is the interpretation intended by the argument. But we can think of it as like the blurred photograph of something in motion. The object is apparently at (many) different places (equal to itself) at the "instant" the photograph was taken.

RP4 If something is at different places during the same instant then it is not at one place equal to itself.

RI5 RP4 => RC3

RC3 If something is at one place equal to itself then it is not at two different places during the same instant.

RI6 RP2 & RC3 => RC4

RC4 An arrow in flight is not at different places during the same instant.

RI7 RC22 & RC4 => RC5

RC5 An arrow in flight does not move.

The "also" clause has more the form of an "otherwise" clause.

RP5 Something cannot occupy a space smaller than itself.

RP6 If something is not at a place equal to itself then it occupies a space either smaller than or larger than itself.

RI8 RP5 & RP6 => RC7

RC7 If something is not at a place equal to itself then it occupies a space larger than itself.

RI9 RP4 & RC6 => RC7

RC7 If something is at different places during the same instant then it occupies a space larger than itself.

RI10 RC22 & RC7 => RC8

RC8 If something moves then it occupies a space larger than itself. (An arrow must occupy a space larger than itself if it is to move.)

RC22, however is also in direct conflict with a widely held premiss.

RP7 Nothing can be in two different places (equal to itself) during the same (one) instant.

This case concludes that the arrow cannot move during an instant. We are left with case 2.

Case 2:

(B) That which moves is at different places (equal to itself) in different instants.

RI41 RC2 => RC21

RC21 Something moves if it is at different places (equal to itself) in different instants. (This is our usual understanding of motion.)

We are considering whether an arrow can be in motion in an instant. By the above case, something could move only if it were in different places in <u>different</u> instants. Therefore, for it to move in the one instant under consideration, that instant would have to have two parts which were also instants. It would be these "sub-instants" in which the arrow were at different places. But, by RD1, an instant is indivisible; so, it has no such parts which are instants. Consequently, *at each instant* it is not possible for the arrow to be at different places in *different instants*.

In either case the arrow cannot move. Consequently the logical disjunction of the two cases also yields an unmoving arrow.

The Stadium

Oppositely marching rows of soldiers pass each other and a standing row of soldiers in the same time. Since oppositely moving rows pass twice as many bodies as each passes stationary bodies, "Zeno concluded that 'double the time is equal to half'".[6] Vlastos states that "Aristotle and all our other ancient informants understood this as a (supposed) paradox of relative motion" and attributes the interpretation which follows to Paul Tannery.[7]

If extension and duration are atomic, that is, there are minimum amounts of each, then an analogy can be made between atoms of extension moving in jumps of atomic time and rows of soldiers drilling in a stadium. Consider three rows of soldiers, one standing (A B C), one marching to the right (1 2 3), and one marching to the left (4 5 6). As the row of soldiers marching to the right passes the row of standing soldiers, it takes one (minimum) unit of time for the soldiers to move one unit of distance -- from a position opposite certain standing soldiers to a position opposite the next ones.

```
1 2 3 =>    After one    => 1 2 3
A B C       time unit      A B C
```

On the other hand, the other row of soldiers, marching left, also move one unit of distance in one unit of time.

```
A B C       After one      A B C
4 5 6 <=    time unit    4 5 6 <=
```

The problem is that the relative change between the soldiers marching left to those marching right is twice the distance in the same amount of time.

```
1 2 3 =>    After one    => 1 2 3
4 5 6 <=    time unit    4 5 6 <=
```

Soldier 6 ends up opposite soldier 1 after 1 unit of time. In order to get there he had to pass soldier 2. This must have taken one unit of time, and passing from there on to soldier 1 must have taken another unit of time. Hence the expression "double the time". It is reasonable to presume that "half the time" refers to the immediately preceding antecedent (the doubled time) rather than to the fixed unit of time which got doubled. Otherwise, we would be looking for a relationship of 4 to 1 instead of 2 to 1. Vlastos confirms this interpretation in his quotation:

> So it follows, he thinks, that half the time equals its double [that $t/2 = t$] (Aristotle, *Physics* 239b35).[8]

It is difficult to get the sense of the conflict because we are accustomed to thinking of time as continuous. The perplexing nature of the situation may be illustrated by noting that there must be some time when soldier 6 is opposite soldier 2, while the argument says that there is not. Unpacking the argument into premisses and conclusions yields the following:

SP1 Soldier 6 passes from soldier C to soldier B in a minimum unit of time.

SP2 The instant at which soldier 6 is opposite soldier C is the instant that soldier 6 is opposite soldier 3, and the instant at which soldier 6 is opposite soldier B is the instant that soldier 6 is opposite soldier 1.

SI1 SP1 & SP2 => SC1

SC1 The minimum unit of time that soldier 6 takes to pass from soldier C to soldier B is the minimum unit of time that soldier 6 takes to pass from soldier 3 to soldier 1.

SP3 If two instants are separated by the minimum unit of time there is no instant between them. (Time is atomic.)

SI2 SC1 & SP2 => SC2

SC2 At one instant soldier 6 is opposite soldier 3 and at the next instant soldier 6 is opposite soldier 1, and there is no instant between these two instants.

SP4 Any soldier passing from soldier 1 to soldier 3 must pass all those in between.

SI3 SC2 & SP4 => SC3

SC3 Because soldier 2 is between soldiers 1 and 3, soldier 6 must pass soldier 2.

SP5 If soldier 6 passes soldier 2 there must be an instant at which this happens.

SI4 SC3 & SP5 => SC4

SC4 There is an instant at which soldier 6 passes soldier 2.

SP6 If there is such an instant, it must be between the instant that soldier 6 is opposite soldier 3 and the instant that soldier 6 is opposite soldier 1.

SI5 SC4 & SP6 => SC5

SC5 The instant at which soldier 6 passes soldier 2 is between the instant that soldier 6 is opposite soldier 3 and the instant that soldier 6 is opposite soldier 1.

SC6 SC5 contradicts SC2

The paradox of plurality

This paradox can be tersely stated as follows: Ultimate parts must have no magnitude or they would not be ultimate parts. But an extended object cannot be made up of parts with no magnitude. Parts of zero size add up to zero size. So an extended object must be "so small as to have no magnitude". The parts must have magnitude. But an infinity of extended parts must have infinite extension. So an extended object must be "so large as to be infinite".[9]

Expanding this statement to show more fully the premisses, inferences, and conclusions yields:

PP1 The size of an extended object is not zero.

PP2 Extended objects are made up of parts.

PP3 Ultimate parts have no magnitude (zero size).

PP4 The number of the parts an extended object is infinite.

PP5 If something is made up of parts then its size is the sum of the size of its parts.

PP6 Parts of zero size add up to zero size.

PP7 An infinity of parts of non-zero size adds up to infinite size.

If an extended object has parts, there are two cases to consider: the parts are ultimate or the parts are extended.

PI1 PP2 > PC1 OR PC2

PC1 An extended object is made up of ultimate parts.

PC2 An extended object is made up of extended parts.

Case 1: The parts are ultimate (PC1 holds).

PC1 An extended object is made up of ultimate parts.

PI2 PC1 & PP3 => PC3

PC3 An extended object is made up of parts of zero size.

PI3 PC3 & PP5 => PC4

PC4 The size of an extended object is the sum of parts of zero size.

PI4 PC4 & PP6 => PC5

PC5 The size of an extended object is zero size. ("It is so small as to have no magnitude").

Case 1 concludes that an extended object has <u>no</u> size -- a clearly unacceptable result. Case 2 fairs no better.

Case 2: The parts are extended (PC2 holds).

PC2 An extended object is made up of extended parts.

PI5 PC2 & PP1 => PC6

PC6 An extended object is made up of parts of non-zero size.

PI6 PC6 & PP5 => PC7

PC7 The size of an extended object is the sum of parts of non-zero size.

PI7 PC7 & PP4 => PC8

PC8 The size of an extended object is the sum of an infinite number of parts with non-zero size.

PI8 PC8 & PP7 => PC9

PC9 The size of an extended object is infinite size. (It must be "so large as to be infinite".)

Case 2 concludes that an extended object must be infinite in size -- an equally unacceptable result.

The paradox lies in the following: it cannot be denied that things are made up of parts; but if things are made up of parts then there are two possibilities, and both possibilities lead to absurd conclusions.

Notes and References

1. Salmon, Wesley, Zeno's Paradoxes, Bobbs-Merrill, 1970, pp. 8-9

2. Salmon, p. 9

3. Salmon, pp. 9-10

4. Salmon, pp. 10-11

5. Edwards, Paul, and others, eds., The Encyclopedia of Philosophy vol. 8, (New York: MacMillan, 1967), s.v. "Zeno of Elea", by Gregory Vlastos, pp. 374-5.

6. Salmon, pp. 11-12

7. Vlastos, p. 375.

8. Vlastos, pp. 369-379.

9. Salmon, p. 14

CHAPTER III

DISCUSSION OF ZENO'S PARADOXES

Salmon suggests that Zeno is not just putting paradoxes forth.

> It has been suggested, and Owen elaborates this theme, that Zeno's arguments fit into an overall pattern. "Achilles and the tortoise" and "The dichotomy" are designed to refute the doctrine that space and time are continuous, while "The arrow" and "The stadium" are intended to refute the view that space and time have an atomic structure.[1]

The pattern suggested by Owen suggests a Zen koan -- a parable with illustrative value that is often logically inconsistent. Such parables are used to stimulate a disciple toward certain realizations, and a perennial theme of Zen koans is that obvious choices should be rejected. In this case the choice is between atomism and infinite divisibility. Heraclitus rejected such a choice in his doctrine of the unity of opposites. Zeno's arguments, purporting to refute both choices, seem compatible with Heraclitus. But, as with Zen, many disciples never master the teachings; some write long dissertations arguing pro or con regarding some illustrative parable. Regarding oneness or manyness -- atomism or infinite divisibility -- many such dissertations have been written.

Zeno confuses discrete and dense sets, and he assumes that zero times infinity is zero. These errors figure prominently in his arguments. In a sense, 'discrete' means atomic and 'dense' means divisible. (See page 100.) Today we are accustomed to distinguishing between these two kinds of sets. Certain properties or attributes are associated with each kind of set. For example, the notion expressed by 'next' or 'successor' is one from counting and is associated with discrete (atomic) sets. Whenever one asks about the next one of any sequence, one is introducing the atomic perspective into the discussion. We have mathematical models of both kinds of sets. The integers are discrete; the rational (and real) numbers are dense. Both numbering systems are applied to extension, as Aristotle notes. (See page 91.) Metric space theory allows discrete as well as continuous measures of extension. But continuous metrics are far more well known than discrete metrics.

<u>On The Achilles</u>

In "Achilles and the Tortoise", Max Black suggests that the paradox derives from an imprecise use of language.

> [The] fallacy in Zeno's argument is due to the use of the words 'never' and 'always'."[2]

But Black does not expand and make explicit his assertion. Instead, he suggests that an infinite series of acts is self-contradictory. Black is guilty of exactly the charge he levies, only with a different word. It is not the infinite series of acts that is problematical; it the notion of *completing* an infinite series of acts on a one-at-a-time basis. Black's statement, "the machine comes to a halt", suggests this sense.[3] The sense of *completion* involving a one-at-a-time process presumes that there is an end to the process, whereas Peano's successor axiom presumes that there is no end to the process (of counting natural numbers). If *complete* is understood in this sense, then "completing" an infinite series of tasks means "coming to the end of a sequence with no end", which is indeed self-contradictory.

> For it is the very essence of [an infinite] progression not to have a last term and not to be completable in that ordinal sense! To maintain the self-contradictory proposition that in such an actually infinite aggregate of order type ω, there is a "last" set of divisions which ensure the completability of the process of "infinite division" by "reaching" a "final" product of division is indeed to commit the Bernoullian fallacy.[4]

But there is another sense of *complete* which must be used while scrupulously avoiding the sense which connotes coming to an end.[5] That sense is expressed by "all there". For finite sets, and in ordinary usage, both senses apply at the same time. In fact, one discovers whether the silverware is "all there" by counting it and coming to the (right) end. When we distinguish between these two senses of complete, we see that we cannot apply both senses to an infinite set. An infinite series of acts, or an infinite set of numbers, is complete in the sense that all acts, or numbers, are included, while it simultaneously fails to be complete in the sense of coming to an end. Finite sets may be distinguished from infinite sets by examining whether the sense of "complete" expressed by 'coming to an end' applies. Black fails to make this distinction.

If we follow-up on Black's imprecise lead we can see that 'always' is used in a particular way in Zeno's argument. It can be properly represented as meaning "for all N". In the form of the argument expressed by Black, Achilles runs at the speed of 10 yards per second, while the tortoise runs at 1 yard per second and is given a 100 yard head start. (Achilles has a 100 yard handicap.) If Achilles runs at 10 yards per second and the tortoise has a 100 yard head start, then Achilles will take 10 seconds to arrive at the spot vacated by the tortoise. (100 Yd /10 Yps = 10 sec.) In that 10 seconds, the tortoise, who runs at 1 yard per second, will have run 10 yards. So, according to this form of the paradox, while Achilles runs the 100 yard handicap, the tortoise has run another 10 yards. This puts the tortoise 10 yards ahead of Achilles. We can think of the race as starting again with a 10 yard handicap. While Zeno runs the 10 yards the tortoise runs one yard.

Let us assign N=1 to the first race and N=2 to the second race. Naturally, there is a third race, and a fourth, etc., etc. In the first race Achilles runs 100 yards; in the second race he runs 10 yards; and in the third race he runs 1 yard. This sequence can be expressed in exponential form as 10^2, 10^1, 10^0, ... The relationship between the distance and the race number can be more clearly seen if we express the sequence as $10^{(3-1)}$, $10^{(3-2)}$, $10^{(3-3)}$, ... We can compute the distance run in each race by Achilles if we use the expression $10^{(3-N)}$, where N is the race number. The tortoise, on the other hand, travels at 1/10 the speed of Achilles. In the 10 second first race he travels 10 yards; in the 1 second second race he travels 1 yard; in the .1 second third race he travels .1 yard. The corresponding sequence for the tortoise is $10^{(2-1)}$, $10^{(2-2)}$, $10^{(2-3)}$, ... We can compute the distance run in each race by the tortoise if we use the expression $10^{(2-N)}$.

By the same reasoning we can discover an expression for the distance between Achilles and the tortoise after each race. After the first race the tortoise is ahead by 10 yards; after the second race by 1 yard, and so forth. Each term in the sequence, 10^1, 10^0, 10^{-1}, . . . , can be computed using the expression $10^{(2-N)}$. This is the same as the amount the tortoise runs, and this makes sense because it is the amount the tortoise runs that puts him ahead of Achilles. Now, it is quite clear that this number is a positive quantity for every N. Since there is never an end to numbers, it is usually concluded that Achilles can never come to the end of the races, that is, Achilles can never catch the tortoise. That there is never an end to numbers is perhaps better stated by Peano's successor axiom: every number has a successor. In the present context we should not conclude that Achilles can never catch the tortoise; rather, we should conclude that for every N, the tortoise has a $10^{(3-N)}$ yard lead on Achilles." While this can be paraphrased as: "the tortoise is 'always' ahead of Achilles" and subsequently re-paraphrased as "Achilles can 'never' catch the tortoise", it is usually forgotten that the 'always' and 'never' are limited by the original context to the integers which index the successive races. The more general sense of 'always' -- "for all time" ('never' -- "at no time") dominates our understanding of the final paraphrase. The argument, in effect, equivocates between two meanings of 'always' or 'never'.

Were we to 'extend' the integers by adding the axiom of infinity, without explaining how Achilles might "complete" the finite races, trans-finite races would place Achilles at, and beyond, the location of the tortoise. But the axiom of infinity essentially assumes that the number infinity exists. In the context of the Achilles, that is tantamount to assuming that Achilles catches the tortoise.

There is a subtle fallacy at work in the argument. It can be illustrated by an analogy. Gödel's incompleteness theorem showed that arithmetic is essentially incomplete. That is, true statements can be constructed in arithmetic which are formally undecidable on the basis of the given axioms. While an undecidable statement may be added as an axiom, the extended system so created is also incomplete.[6] As noted above, "nothing in these premisses allows us to conclude that Achilles actually catches the tortoise." The statement "Achilles catches the tortoise", is not decidable on the basis of the premisses and inferences given for The Achilles. It is, of course, true that Achilles catches the tortoise. Like Gödel's constructed undecidable statement, "Achilles catches the tortoise" may be assumed true and added as a premiss.

AP10 Achilles catches the tortoise.

And, by the earlier mathematical analysis, this corresponds to assuming the axiom of infinity. The truth of the premiss is demonstrable by means other than the system of premisses and inferences, so we cannot argue that the premiss could be assumed to be false. We are without a *means* to demonstrate its truth, and that is an epistemological issue. The fallacy is in confusing epistemological and metaphysical issues. We may not infer that, because we can see no way to infer the truth of a statement from certain premisses, the statement is therefore false. But this seems to be what is happening in the Achilles. Of course, this fallacy is being facilitated by equivocation between different senses of 'always' ('never'). This effectively disposes of "the Achilles" as a paradox.

On The Dichotomy

The first form of the Dichotomy suffers from the same fallacy as the Achilles. The computation may be made somewhat simple by presuming that the total distance to be 1 unit. When Achilles has reached the half way point, his distance traveled is 1/2 the total, and the remaining distance is 1/2 the total. Then, when he has reached half the rest of the way, his distance traveled is the original 1/2 plus 1/2 the remaining 1/2, or $1/2 + 1/2 \cdot 1/2$, and the remaining distance is 1/2 the previously remaining 1/2, or

1/2•1/2. We can safely assume that Achilles, being "the fleetest of all Greek warriors"[7], races toward the unreachable point, and construct a table showing his progress after each "race".

Race #	Distance traveled	Remaining distance.
1	1/2	1/2
2	1/2 +1/2•1/2	1/2•1/2
3	1/2 +1/2•1/2 +1/2•1/2•1/2	1/2•1/2•1/2
4	1/2 +1/2•1/2 +1/2•1/2•1/2 + /2•1/2•1/2•1/2	1/2•1/2•1/2•1/2

The distance traveled for each race after N races can be expressed as the series: $(1/2)^1$, $(1/2)^2$, $(1/2)^3$, $(1/2)^4$, . . . , $(1/2)^N$, and the remaining distance can be expressed as: $(1/2)^N$. The fraction Achilles has already covered after N races is just the sum of the first N terms of the above series and may be expressed as $\sum_{I=1}^{N}(1/2)^I$. This sums to 1-$(1/2)^N$. The proper conclusion is "For all natural numbers N, Achilles has a $(1/2)^N$ fraction of a unit remaining to run". As in The Achilles, that Achilles has a positive fraction remaining to run *for all N* does not mean that he has a positive fraction to run *for all time*.

A New Resolution of the Paradoxes

There is yet another resolution of the paradoxes. The proposed resolution is one that I have found no mention of in any of the writings. The new resolution can be approached with an example from model-theoretic semantics. A *model* consists of a language, a set of objects, and an interpretation function from the language to the objects, sets of objects, and relations among the objects. It is not necessary to go into the technical structure of models in more detail to present the formal structure of the proposed resolution.

Suppose there are two such models related in a particular way. The two languages are different, and the set of objects from one model is a proper subset of the objects from the other model. For the purposes of this discussion I shall refer to the model whose objects are the proper subset as the *limited* model; I shall refer to the other one as the *general* model. I shall also use these terms to refer to the respective parts of the models.

In such a structure it is possible to use a set or sequence of terms from the limited language to pick out an associated set or sequence of limited objects. Since these objects are also members of the objects in the general model, a corresponding sequences of terms in the general language can also be constructed. Because the languages are different, there is nothing to suppose that there cannot be additional objects, in the general model and not in the limited model, which may be selected by the terms of the general sequence.[8] By way of an analogy we may correlate colloquial language with formal language, and reference to the interpretation function. Under this analogy, when a formal language term picks out some object or set of objects, the colloquial language refers to or *describes* the analogous object.

In the first form of the dichotomy, Achilles' inexorable dash achieves and passes the limit, not by following the sequence as N gets bigger (limited language), but by remaining in motion for a sufficient time (general language). The distance covered is the product of the velocity and the time elapsed. Under my new resolution the first form of the dichotomy and the elapsed time argument are both *descriptions* of the physical race. It is possible to *describe* Achilles' position using the infinite series method only up to the limit of that series. The language of the series cannot describe what happens at or beyond its limit. The description of the race using linear velocity, time, and distance can describe what happens at and beyond the limit of the series. Since the infinite series or limited system of representation cannot describe events past its limit, its use should be suspect.

The general fallacy implicit in the arguments seems to be the belief that the totality of all languages can describe everything -- that nothing can exist that language cannot describe -- a point not universally accepted. Colloquial languages can describe things which do not exist, and we are accustomed to describing only those things the language can describe. Like the blind-spot, the contents of which we do not see, and the existence of which we are normally unaware, languages have limits; they do not describe some things, and we are normally unaware that indescribable things may exist. It is quite clear that limited subsets of language cannot describe everything the whole of language can. That a particular subset of language, which we might call a system of description, cannot describe certain objects does not entail that those objects cannot exist. Yet that is just what both the Achilles and the Dichotomy would have us accept.

The second form of the dichotomy suffers from equivocation on the sense of 'complete' as well. The idea that Achilles can't even get started comes from the view that his first step must be onto the "last" member of an infinite sequence of bisections. If the sequence is "complete" in the sense of coming to an end, then Achilles can step onto that end for his first step. But no such end exists, so Achilles has no place to make his first step. This view requires that the members of the series and the starting point stand in the successor relation to each other. But the limit of a sequence does not stand in the successor relation to any of its members. $\aleph_0$ has no predecessor.[9] [See the discussion at the bottom of page 60.] The limit of the sequence stands in a relation to its members not unlike that of the relationship of a pedestal to the floor around it. Whenever Achilles steps off the pedestal down onto the series, he has already stepped past an infinite number of elements -- however small his step. Similarly, he cannot get back on the pedestal by following the successive steps; he must jump up through an infinite number of steps onto the pedestal (the first form of The Dichotomy).

With the pedestal analogy in mind, the second form of the Dichotomy seems not to pose a problem as long as extension can continually be divided in half. If it can, then the conclusion seems acceptable. Nothing would be wrong with "covering" an infinite number of points. But the conclusion depends upon the presumption that extension is, in fact, infinitely divisible by bisection. If, at any stage in the process, half way becomes an indivisible minimum, as would be the case were extension not infinitely divisible, then getting half that way becomes an impossible act. In other words, this argument begs the question by presuming that extension is infinitely divisible. It is supposed to prove atomism by concluding that an infinite number of points must be covered -- on the premiss that covering an infinite number of points is an impossible act. But there are an infinite number of points to cover <u>only if extension is infinitely divisible</u>; the argument contains a premiss which presumes the purported result. One cannot cover half the minimum distance. <u>The premiss "for Achilles to reach any point he must get half way" is not true when the point he must reach is only the minimum distance away.</u> He must get there in one fell swoop since there is no intermediate "half-way" point. That he can cover an infinite number of points entails the presumption of divisibility.

J. O. Wisdom noticed the difficulty in "Achilles on a Physical Racecourse", but he interpreted it as supporting his contention that the argument is intended to apply to theoretical divisibility.

> This, I think, is the easiest way of seeing that Zeno's premiss cannot characterize a physical race: the 'and so on' is inapplicable because somewhere two neighboring physical points will touch each other and it will be impossible to subdivide the distance between them without altering the assigned size of the points. [sic][10]

<u>On both the Achilles and the Dichotomy</u>

James Thomson attempts to resolve the problems of infinite divisibility by appealing to the definition of "infinite" in the sense of arbitrarily large. Any such number chosen is still finite. He distinguished this sense from the sense in which "infinite" is taken to mean an unending process and from the sense in which infinite means the first number which every finite number is less than ($\aleph_0$).

> "[T]o say that a lump is infinitely divisible is just to say that it can be cut into any number of parts. Since there are an infinite number of numbers, we could say: there is an infinite number of numbers of parts into which the lump can be divided. And this is not to say that it can be divided into an infinite number of parts. If something is infinitely divisible, and you are to say into how many parts it shall be divided, you have $\aleph_0$ alternatives from which to choose. This is not to say that $\aleph_0$ is one of them."[11]

Thomson proposes that it is the infinite number of ways that an object may be divided that unpacks what "infinitely divisible" means. In any such way of dividing it, the object would actually be divided into only a finite number of extended parts. This alternative avoids the obviously contradictory notion of completing an unending process of repeated division. This is not, however, the argument Zeno advanced, and Thomson defines the notion of "super-task" for the purpose of resolving Zeno's argument.

Thomson defines a *super-task* as a task that is completed if and only if an infinite number of tasks are completed. To suggest that an infinitely divisible object is actually divided in this way would be to presume that an unending sequence of divisions had been completed. The presence of both "unending" and "completed" in the reformulation of this premiss makes it rather obviously self-contradictory (if 'complete' is taken to mean 'came to the end').

It is just this apparently self-contradictory nature which determines the actual formulation for the axiom of infinity. In terms of ordinary numbers, the axiom can be stated: there is a number X (infinity) such that 1<X and whenever N<X then N+1<X.[12] By formulating the axiom in this manner, one makes no attempt to "complete" the process of "+1" in any way whatsoever; one merely conforms to the requirements of mathematical induction. We may use induction to infer that for all N, N<X.

Notice that X is not the successor of any number N because N+1<X. There is no number N whose successor is X; consequently, X does not have a predecessor[13]. Thomson alludes to this point, but is not explicit at all.

> But it is obviously unreasonable to ask where the runner was when he was at *the point immediately preceding his destination*. [Italics mine][14]

One reason it is unreasonable is that there is no such point; that is, $\aleph_0$ has no predecessor.

If the infinite sequence of remaining halves has been traversed and one is at the point 1, this is equivalent to having ascended to the number $\aleph_0$. But $\aleph_0$ has no predecessor so there is no immediately preceding point to have been at. Most people are more familiar with decimal mathematics, so the equivalence may be better illustrated by taking the fraction traversed (in the Dichotomy) to be 9/10ths rather than one half. The first step covers 9/10ths (.9) of the way; 1/10th remains. The second step covers 9/10ths of the remaining 1/10th (9/10·1/10) or .09, and the total distance traversed after two steps is .9 + .09 = .99. The next step covers .009 for a total of .999. "Completing" the "super-task" mentioned above requires traversing .999999..., and it can be shown that that is just being at 1.

The following demonstration shows that the number expressed by .999999... is the same number as is expressed by 1. If these are the same number, then to have traversed *all* the remaining 9/10ths is just to have reached 1.

<u>Proof of .99999... = 1.</u>

(1) Let	X = .99999...
(2) Then	10X = 9.99999...
and (2)-(1) is	10X - X = 9.00000...
Therefore:	9X = 9
and, by division,	X = 1.

Completing the sequence of repeatedly getting 9/10ths of the rest of the way just *is* being at 1. In other words, if one has completed the sequence then one is at 1. So, by modus tollens, if one is not at 1, one cannot have completed the sequence. Consequently, one cannot have completed the sequence without being at 1. To presume that one has completed all the points in the sequence but is not at 1 is to presume a hypothesis that is false -- which leads, of course, to anything at all.

Owen seems aware of this, but violates it in the same breath that he denies doing so.

> Notice that Zeno is not first setting up a division which cannot have a last move and then asking, improperly, what the last move would be. He is asking, legitimately, what the total outcome of the division would be; and for there to be such an outcome there must be a smallest part or parts.[15]

The size of the parts is bounded below. In fact, there is a greatest lower bound, and it is zero. However, this greatest lower bound is not *in* the set of sizes. Consequently, contrary to the above claim, there is no smallest part or parts. Because there is no smallest part we cannot conclude that the size of any such parts resulting from the division process is zero. Mathematical induction shows that it is not. The size of the parts is bounded below by zero, but no part is of size zero, and there is no smallest part. The real numbers under the usual order relation are not "well-ordered". If they were well-ordered, then the greatest lower bound would be in the set, and there would be a smallest (last) member whose size would indeed be zero.

On The Arrow

The physics of the relation between position and velocity has some interesting structural consequences. The Heisenberg Uncertainty Principle states that the position and the momentum (velocity) of an object cannot be simultaneously measured to any degree of accuracy; accuracy in the measurement of one is lost at the expense of accuracy in the measurement of the other[16]. A homely macroscopic analogy illustrates this principle.

Take a photograph of an object in motion. The length of time the shutter is open (the reciprocal of the shutter speed) can be used in conjunction with the amount of blur in the image to estimate the speed of the object. The longer the shutter is open the longer the blur and the more accurately the speed can be measured. But the longer the blur is, the less accurately one is able to determine the position of the object. Conversely, the sharper the picture is, the more accurate knowledge of the position of the object will be, but the more uncertain knowledge of its velocity will be.

In Zeno's thought experiment a very sharp view, namely, the arrow being "in its place", is taken; this leaves no blur at all to use in determining the velocity. The mental shutter speed would have to be infinite to obtain an indivisible instant -- we are left with an instant with zero duration. Since motion is measured by determining the ratio of distance traveled to the time duration, Zeno's thought experiment leaves zero (length blur) divided by zero (length duration) for the computation of velocity. And zero divided by zero is undefined. One has perfect information about the position but no information about the velocity. While it is true that a stopped object leaves no blur, it seems fallacious to assume that velocity is zero when one sees no blur. And any blur at all does have the immediate consequence that the object is indeed occupying a space larger than itself.

It has been argued that an object always takes up the space it occupies. It can also be argued that an object in motion always takes up <u>less</u> space than it occupies. Relativity theory holds that an object in motion is contracted in the direction of motion. The shortened length, X', can be calculated in terms of the at-rest length, X, and the velocity, V, using the Lorentz contraction equation. That equation is $X'=X\cdot(1-V^2/C^2)^{1/2}$, where 'C' is the speed of light.[17] The faster an object is moving, that is, the larger V is, the smaller X′ is. If an instant is indivisible and is the same "size" regardless of whether the object is at rest or in motion, then the moving, and hence contracted, object takes up less space; it has room to rattle around in the same sized instant which immobilizes the object at rest. But according to relativity theory objects in motion experience a time dilation effect. That can be interpreted to mean that the size of an instant is increased or "stretched".[18] This gives the moving object even more room. Also according to relativity theory faster moving objects are contracted more. The greater the contraction, the relatively greater the room to move -- and hence the greater the speed possible.

Consider the possibility that an instant is indivisible in the sense that it has no duration. An instant without duration is not consistent with our usual definition of velocity. Velocity is the ratio of distance traversed to the duration in which the traversal occurs. An instant with no duration would be just the temporal coordinate of an object. Taken together with its position coordinate, the result forms the event coordinates of a particular space-time point. It is not possible to determine the velocity of an object on the basis of a single event. At least two events are required. Even our notion of velocity at a point depends upon more than one event. We define the velocity at a point as the instantaneous rate of change of position with respect to time. An instantaneous rate of change, from the differential calculus, involves taking the limit of the ratio of the change in position (distance) to the change in time (duration) <u>between two points</u>. That limit may be some definite quantity, but all computations not at the limit require an extended duration, and no computation is possible *at* the limit (the denominator would be zero, and division by zero is forbidden). If an instant has no duration, then no velocity *within the instant* is possible. Velocity at an instant is not determinate without reference to the context of the instant, that is, events external to the instant.

The renditions of the argument as I have presented them significantly reflect our modern view of motion. The Arrow as originally presented speaks only to motion within an instant. The definition of motion I proposed, a modern one, in conjunction with the attending argument concludes, validly, that motion is a phenomenon that is "trans-instantaneous". If instants are infinitely divisible, as the divisionists presume, then any moving object exists in different places (equal to itself) at different instants, as close as you like. On the other hand, If instants are atomic, as the atomists presume, then any moving object exists in different places (equal to itself) at different instants, and the motion is as described more fully in the discussion of "uniform" velocity on page 72 below. But in both cases the concept of moving, as we understand it, cannot apply "within" an instant. Our concept of motion, which is based upon infinite divisibility, requires that we think of objects as moving continuously. If we think of instants as intervals (durations), that means we think of the object as crossing each interval continuously until it enters the next instant, crossing that into its successor, etc. But this view is not consistent with atomism, as the Arrow shows.

On The Stadium

The alternative to an instant with no duration is an instant which is indivisible in the sense that it is atomic, that is, that it has a minimum but indivisible duration. It is exactly this premiss that is used in the argument known as *The Stadium*.

The stadium purports to prove that objects may pass each other without ever being opposite one another. That conclusion is presumed to be absurd and stands as the basis for seeing The Stadium as a paradox. But we seem to have actually observed just such behavior in the context of quantum mechanics, and that observed physical behavior may be offered as an empirically based counter-example to The Stadium. Protons appear to "by-pass" a point where they cannot physically exist in the physical process known as tunneling.[19] A proton in a radioactive nucleus having enough energy to exist outside the nucleus is confined to the nucleus by a potential field with an annular region requiring a greater potential than the energy of that proton. Because the proton has less energy than is required to be in this area, the area is called a potential barrier. This barrier confines the proton to the region of the nucleus of the atom. There is also a region outside the potential barrier in which the potential is less than the proton requires to be there. For a proton to traverse from the inner region to the outer region, it must pass through a region requiring more energy of its occupants than the proton has. The probability of the proton being located there is therefore zero, but the observation is that some protons do get out.[20] They do so without being in the restricted region.[21] They are described as having "tunneled"[22] through the potential barrier.[23] In doing so they, in effect, "march" past others without ever having been opposite them.

Relativity theory offers us another model that is incompatible with our "common sense" experience. Relative velocities do not add according to ordinary arithmetic. The relative velocity of the row of soldiers marching left to the row of soldiers marching right (V3) is not just the sum of the relative velocity of the row of soldiers marching left to the row of soldiers standing (V1) plus the relative velocity of the row of soldiers standing to the row of soldiers marching right (V2). In the case of very high velocities the sum can be nearly the same as the original velocities. The formula for the resulting velocity -- the law of addition of velocity -- in terms of ordinary addition, is: $V_3 = \frac{V_1 + V_2}{1 + V_1 . V_2}$, where all velocities are expressed as a fraction of the velocity of light (C).[24]

Suppose our soldiers are marching <u>very</u> fast. If the relative velocity of the moving soldiers to the standing ones is nine-tenths the speed of light (.9C) then the relative velocity between the two oppositely marching rows of soldiers is not twice this (.9C + .9C = 1.8C); it is only $\frac{.9C + .9C}{1 + .9C..9C} = .9945C$.

In addition, objects moving at relativistic speeds experience a contraction or foreshortening in the direction of travel. This contraction means that the minimum length becomes smaller as the velocity increases. The amount of this foreshortening is computed using the Lorentz contraction formula -- $X' = X(1 - \beta^2)^{1/2}$, where $\beta = V/C$.[25] In the case of the soldiers marching at .9C relative to the standing ones, the contraction computes to $(1 - .9^2)^{1/2} = .4359$. This means that two of the moving soldiers take up less space than one standing one -- with room to spare! This would mean that the moving atomic units would take up enough <u>less</u> space that both could be opposite the one stationary atomic unit at the same instant. As the fast moving soldiers pass the standing ones, the minimum distance in the direction of travel has shrunk by over a factor of two to one. This, paradoxically, would allow two moving soldiers to be opposite one standing one! Since this is true of both rows, a standing soldier would see two moving ones on each side, and they would appear to be passing one another.

In the case of the soldiers marching at .9945C relative to the row of oppositely marching soldiers, the contraction computes to $(1 - .9945^2)^{1/2} = .1050$. If one takes a "stand" with one row of moving soldiers, the relative contraction of the other moving soldiers is nearly a factor of 10 to 1. This would allow nearly 10 soldiers to be opposite one! But relativity has its own paradoxes. To the other moving soldiers, it would also seem that ten were opposite one.

There is other evidence that calls into question the argument of the stadium -- evidence that does not need to evoke relativistic speeds. This particular evidence comes from the studies of human perception. As such it represents an epistemological approach. We believe that we can perceive smoothly continuous motion and that that smooth perception would contradict the atomistic assumptions about time and space. However, psychological experiments show that velocity perceived as smooth may in reality not be smooth. In the experiment an observer sees two distinct lights. One is turned off and the other is turned on in sequence. Within a narrow range of the parameters of separation and the duration in which both lights are extinguished, the two lights appear to observers as one light which moves smoothly instead of as two lights which blink off and on respectively. The underlying neurological processes involve the length of time an image of a stimulus is retained in the neural circuits and our experience with moving objects. Under certain conditions we cannot perceptually distinguish between a continuously moving object and one which ceases to exist at one point and then begins to exist at another point (as would be the case under the hypothesis of atomism). An argument from the perception of smooth motion that atomism is false fails because perceptual experiments show that smooth perception arises in response to atomistic jumps in motion.

About both the arrow and the stadium

It seems clear that the various writers recognize that movement involves a change of position over different parts of a span of time. But the two arguments, the arrow and the stadium, seem to presume that motion is somehow continuous. Continuous motion would require continuous space and time. The arguments suggest that a contradiction results from assuming that space and time have an atomic structure. The contradiction actually results from the implicit assumption that movement does not also have an atomic structure. Were movement to also occur in discrete units, no contradiction would arise. A discrete movement would occur when an arrow was in one position at one instant of time and in another position in another instant of time (like electrons changing energy levels in "atoms"). Clearly, the slowest non-zero "uniform" velocity occurs when the arrow occupies the adjacent position during the successive instant. A slower motion would require the object to remain "at rest" in the same position during more than one successive instant. Such motion would be full of starts and stops. However, a faster motion would require "skipping" positions. Moving twice as fast would skip every other position. But it is the assumption that motion is continuous that is required to show that an object must also pass, or be in, any intermediate position. The mean value theorem in calculus proves that, in the case of a continuous, monotonic increasing function, the midpoint must be passed by the function at some intermediate point. A key prerequisite for this theorem is that the function concerned be continuous.[26] It is similarly so with the paradoxes of the arrow and the stadium. For the arrow which moves twice as fast to occupy the intermediate position, it must also occupy an intermediate instant. But by assumption there is no intermediate instant -- because time is assumed to be atomic and not continuous. Motion cannot be continuous when time and space are not. Assuming that motion is continuous assumes a contradictory premiss and anything follows.

On The Paradox of Plurality

If something is composed of parts then the parts must also be composed of parts, ad infinitum. This leaves open the question "what is a part?" To give a definition of "part" in the context of the plurality hypothesis, one must necessarily produce a circular definition. "A *part* is that which is composed of parts." In modern times we could salvage this circularity by making the definition recursive.[27] "A *part* is either an *ultimate part*, or it is composed of parts." Of course, for the definition to be satisfactorily recursive, it must provide some reassurance that parts are composed of ultimate parts after a non-infinite number of reentries. It is, after all, the finite number of reentries that distinguishes recursion from circularity and infinite regress. The plurality hypothesis, as it is represented, entails a circular definition of *part* just because there is no base case definition of a part nor any way to terminate the regression back to such a base case after a finite number of tries. A part is a part is a part is a part ... can continue infinitely.

Many of the premisses no longer command the loyalty they once did. There are many counter-examples that cast doubt on the truth of the premisses. I shall examine each statement which forms a part of the argument (as presented on page 41) and present one or more examples where the premiss entailed by the statement no longer holds.

Consider the first statement, "Ultimate parts must have no magnitude or they would not be parts." This statement just defines what an ultimate part is. An *ultimate part* is just a part with no magnitude. The second statement, "But an extended object cannot be made up of parts with no magnitude", is questionable in the light of modern physical theory. According to the standard model in modern physics, the extended proton is composed of a finite number (3) of unextended quarks. The proton consists of two up and one down quarks.[28]

> The proton and neutron are both about 10^-13 cm, or about 1/100,000 of the size of an atom. . . . By indirect means the sizes of quarks and electrons are known to be less than 10^-16 cm, less than 1/1000 of the size of neutron or proton. Indeed, there is no evidence that these particles have any size at all, they may be thought of as points of matter occupying no space.[29]

This clearly shows that the third statement, "A finite number of unextended parts cannot comprise an extended whole", is not true for particle physics. While, for practical purposes, the premiss is almost always true, it is a fallacy to apply it by analogy to sub-atomic particles.

Modern mathematics also permits questioning this premiss, as Grünbaum notes.

> In the context of modern mathematics, Zeno is thus defying us to obtain a result differing from zero upon adding all the lengths of the super-denumerable infinity of points that compose a unit segment. This means that we are being asked to add as many zeros. To Zeno's mind, it was axiomatic (1) that such an addition is necessarily feasible and permissible and (2) that the result of any addition of zeros would be zero, regardless of the cardinality of the set of zeros to be added. But he could not anticipate that the addition of a super-denumerable infinity of numbers, be they zero or positive, presents a problem altogether different from adding *either* a finite sequence of numbers such as 3, 4, 7 *or* a *denumerable* infinity of numbers such as 1, 1/2, 1/4, 1/8, 1/16, 1/32, ...[30]

There are two models from mathematics. The easier model is from integral calculus and is the definite integral, which involves denumerable infinity. The definite integral of a function computes the area under the curve between two limit points. Such a function is integrated by dividing the area under the curve into a finite number of pieces, summing the area of the parts, and then by taking the limit of the sum as the number of parts gets larger and the parts themselves get smaller. The limit of the number of the parts is infinite. The limit of the size of each part is zero.

The sum over these parts can be non-zero and non-infinite. For a simple example, I will show the steps involved in integrating the function X^2 over the interval from 0 to 1 -- $\int_0^1 X^2dx$. While there are many different methods for approximating the area under the curve, the simplest involves treating each segment as a small rectangle. First, the interval is divided into N pieces. The width of each part is 1/N, but the height depends upon which rectangle (the I^{th}) is considered. Since the function X^2 always increases, the right-most point of each rectangle is the highest point in the interval. If we call it the I^{th} rectangle (out of N of them), that height is $(I/N)^2$. The area of the I^{th} rectangle is its width times its height: $(1/N)\cdot(I/N)^2$. The area under the curve is approximated by adding the area of all these small segments. That value is just the sum from I=1 to N of $(1/N)\cdot(I/N)^2$ -- $\sum_{I=1}^{N}(1/N)\cdot(I/N)^2$.

Computing the exact value of the area just involves taking the limit as N approaches infinity. Since the size of each piece is $(1/N)\cdot(I/N)^2$, this limit is zero; and the limit of N is infinity. We have an infinity of parts of size zero, yet the resulting area is neither zero nor infinite. To show that the sum is a definite number we can algebraically manipulate the sum before we take the limit. The expression for the sum can be simplified by factoring out (1/N). This reduces to $(1/N)^3\cdot(\sum_{I=1}^{N} I^2)$. But $\sum_{I=1}^{N} I^2 =$ $N\cdot(N+1)\cdot(2\cdot N+1)/6$[31], so the sum is the product of this and $(1/N)^3$ -- $(1/N)^3\cdot[N\cdot(N+1)\cdot(2\cdot N+1)/6)]$. This multiplies out to $[(2\cdot N^3+3\cdot N^2+N)/6]/N^3$. Simplifying, we get $1/3 + 1/(2\cdot N) + 1/(6\cdot N^2)$, which, as N approaches infinity reduces further to 1/3. So, $\int_0^1 X^2 dx = \lim_{N\to\infty} [1/3 + 1/(2\cdot N) + 1/(6N^2)] = 1/3$, which is clearly neither zero nor infinite.

The second mathematical model involves taking transfinite sums of infinitesimals. These can also be non-zero and non-infinite. Illustrating the model is not necessary, however, as the principle is similar to that of integration.

The statements, "Parts of zero size add up to zero size. So an extended object must be so small as to have no magnitude", commit another fallacy when it comes to the context of an infinitude of parts. Adding the same quantity many times is the same as multiplying that quantity by the number of times it is an addend. For example, 10 added to itself 4 *times* (10 + 10 + 10 + 10) is 10 multiplied by 4. In the immediate context adding an infinitude of zero sizes, that is, adding zero an infinite number of times, is the same as multiplying 0 times infinity. But multiplying zero times infinity is one of the undefined operations, or at best produces an indeterminate result. That zero times anything is zero is a fallacy; zero times infinity is not defined.

The next statement in the argument is: [Therefore,] "the parts must have magnitude." Ordinarily, the argument to this point would constitute a reductio that there can be ultimate parts or that an extended object can be made up of ultimate parts, but the argument proceeds (rhetorically) for effect.

From "the parts must have magnitude" it does not follow that that magnitude must have a non-zero lower limit. Look, for a moment, at the process of bisecting an extended length. For simplicity take the length of the extended object to be 1 unit. When the first bisection is completed the length of each part is 1/2 (half the length of the original). Now consider a part which results when the unit has been bisected N times. Its length is $(1/2)^N$. $(1/2)^N$ is still extended. When we bisect that length the result is also still extended and is of length $(1/2)^{N+1}$. I have just shown that bisecting (dividing) an extended length yields an extended length for N=1 bisections. I have also shown that whenever the length after N bisections is extended, then so is the length after N+1 bisections. These two premisses satisfy the requirements for mathematical induction and we may conclude that "for all N, after N bisections the length is extended". So, the process of bisection yields an infinity of parts, all of which are extended. But notice that after one bisection there are two parts of size 1/2; after two bisections there are 4 parts of size 1/4; and after N bisections there are 2^N parts whose length is $(1/2)^N$. In each case the total length of the parts adds up to the original length.

To assert that an object is composed of unextended parts because this limit of the size of the parts is zero is just not valid reasoning. The flaw is very subtle and has been the basis of controversy in mathematics for millennia.

It confuses the limit of a sequence with the members of that sequence, a point which Thomson notices:

> Hence Whitehead emphasized that the sequence
> 1, 1/2, 1/4, . . .
> was convergent and had a finite sum. He also thereby pointed out a play on the word 'never'; the sequence never reaches 0, the sequence of partial sums never reaches 2. (The sequence does not contain its limit: but it is convergent, the limit exists.)[32]

The argument, "But an infinity of extended parts must have infinite extension", is based upon the unwarranted assumption that because the parts are all extended there must be a smallest non-zero size to the parts. Were there such a limit, then infinite extension would follow. But there being no such limit, infinite extension does not necessarily follow. Some infinite series converge; some diverge.

This argument goes past the point of establishing a contradiction, and hence a reductio; I would think that it does so for the poetic license of being able to say that an object composed of a plurality of parts must be both small and large without limit. The argument above purports to establish the existence of an extended object with no magnitude. It actually establishes that an object made up of parts cannot be made up of ultimate parts. A missing premiss is that any object composed of parts is composed of ultimate parts.

The question that arises is: are ultimate parts composed of parts? The first tendency would be to say no, that that would prevent their having been ultimate. But a second reading is possible. That ultimate parts have no magnitude does not mean that they have no parts. A line has no width but it has both a left and a right side. Moreover, these parts combine to form the whole which has no magnitude (width). An alternate definition of 'ultimate part' could be: "A part is an *ultimate part* if it has no parts." But this alternate definition clashes with the notion that a part is composed of parts ad infinitum. There can be no ultimate parts under the plurality hypothesis if ultimate parts are parts which have no parts.

Notes and References

1. Salmon, Wesley, *Zeno's Paradoxes*, (Indianapolis: Bobbs-Merrill, 1970), p. 12.

2. Max Black, "Achilles and the Tortoise", in *Zeno's Paradoxes*, ed. Wesley Salmon, (Indianapolis: Bobbs-Merrill, 1970), p. 69.

3. Black, p. 74.

4. Adolf Grünbaum, "Zeno's Metrical Paradox of Extension", in *Zeno's Paradoxes*, (Indianapolis: Bobbs-Merrill, 1970), p. 189.

5. It is a developmental enhancement to resolve one concept into two. When the need to do so arises, both concepts usually apply to previously considered cases, but the resolving case necessitates the distinction because one concept applies while the other does not.

6. Ernest Nagle and James R. Newman, *Gödel's Proof*, (New York: New York University Press, 1958), p. 86.

7. Salmon, p. 9.

8. Suppose the general model is $\langle L,I,A\rangle$, the limited model is $\langle l,i,a\rangle$, and $a \subset A$. Let z be a sequence of terms in l. Then $i(z)$ is a subset of a. Let Z be a sequence of terms in L such that $I(Z) = i(z)$. We would represent this as $I^{-1}(i(z))$. Clearly Z is not all L. This would contradict the hypothesis that $a \subset A$. Since $a \subset A$, A-a is not Null. Let X be a sequence in $I^{-1}(A-a)$. If we combine the sequences Z and X, objects will be picked out which include the sequence $i(z)$ as well as additional objects in A-a which cannot be picked out by any sequence in language l. Nothing in the foregoing precludes the possibility that l is a subset of L and i is just I restricted to l.

9. Bertrand Russell, "The Problem of Infinity Considered Historically", in *Zeno's Paradoxes*, ed. Wesley Salmon, (Indianapolis: Bobbs-Merrill, 1970), p. 56.

10. J. O. Wisdom, "Achilles on a Physical Racecourse", in *Zeno's Paradoxes*, ed. Wesley Salmon, (Indianapolis: Bobbs-Merrill, 1970), p. 87.

11. James Thomson, "Tasks and Super-Tasks", in Zeno's Paradoxes, ed. Wesley Salmon, (Indianapolis: Bobbs-Merrill, 1970), p. 91.

12. Keith J. Devlin, Fundamentals of Contemporary Set Theory, (New York: Springer-Verlag, 1979), p. 52.

13. Russell, p. 56.

14. Thomson, p. 100.

15. G. E. L. Owen, "Zeno and the Mathematicians", in Zeno's Paradoxes, ed. Wesley Salmon, (Indianapolis: Bobbs-Merrill, 1970), p. 143.

16. Herbert A. Pohl, Quantum Mechanics for Science and Engineering, (Englewood Cliffs, New Jersey: Prentice-Hall, 1967), pp. 11-12.

17. Edwin F. Taylor and John Archibald Wheeler, Spacetime Physics, (San Francisco: W. H. Freeman and Company, 1963), pp. 64-66.

18. Taylor and Wheeler, p. 66.

19. Pohl, pp. 50-56.

20. While some radioactive atoms decay by exactly this process, others decay by a similar process, but by emitting other particles than protons.

21. There are no known macroscopic models for such behavior -- passing a place without being there.

22. The process is called "tunneling", it seems to me, as a direct result of the perceived absurdity of passing a point without being there and the strength of our commitment to the premiss that objects remain in existence.

23. We have even invented a number of electronic devices that depend upon this theory to work -- (the tunnel diode is one).

24. Taylor and Wheeler, pp. 50-51.

25. Taylor and Wheeler, p. 66.

26. William McGowen Priestly, Calculus: An Historical Approach, (New York: Springer-Verlag, 1979), p. 278.

27. The factorial function, written N!, can serve as an example of a recursive definition. The factorial of a number, N, is the product of that number and all numbers less than it, down to and including 1. 0! is 1 by definition. The above definition can be expressed by a recursive definition as follows:

The factorial of N is defined recursively as follows: If N is zero then the value of N! is one. Otherwise, the value of N factorial is N times N-1 factorial.

$0! = 1, N! = N \cdot (N-1)!.$

If we tried this definition for N = 4, we would get the following sequence:

	$4! = 4 \cdot 3!$
First reentry (N=3)	$3! = 3 \cdot 2!$
Second reentry (N=2)	$2! = 2 \cdot 1!$
Third reentry (N=1)	$1! = 1 \cdot 0!$
Fourth reentry (N=0) (and return)	$0! = 1$
Return from Third	$1! = 1 \cdot 1 = 1$
Return from Second	$2! = 2 \cdot 1 = 2$
Return from First	$3! = 3 \cdot 2! = 3 \cdot 2 = 6$
	$4! = 4 \cdot 3! = 4 \cdot 6 = 24$

Because the number N is decreased at each "reentry", it is guaranteed to reach zero and terminate the sequence after a finite number of times.

28. Robert A. Meyers, ed. <u>Encyclopedia of Physical Science and Technology</u> Vol. 5. (Orlando: Academic Press, 1987), s.v. "Elementary Particle Physics", by Timothy Barklow and Martin Perl, p. 16.

29. Meyers, pp. 13-14.

30. Adolf Grünbaum, "Modern Science and Refutation of the Paradoxes of Zeno", in <u>Zeno's Paradoxes</u>, ed. Wesley Salmon, (Indianapolis: Bobbs-Merrill, 1970), pp. 167-8.

31. William H. Beyer, ed., <u>CRC Standard Mathematical Tables</u>, 25th ed., (West Palm Beach, Florida: CRC Press, 1978), p. 72.

32. Thomson, pp. 101-2.

CHAPTER IV

ARISTOTLE (384-322 BC) AND INFINITE DIVISIBILITY

Before we can reasonably examine Aristotle's views on the subject, we need to briefly outline the events and conditions that transpired between Zeno and Aristotle. It was during this period that true Atomism was born.

The Birth of Atomism Proper

The birth of atomism in its modern form can be traced to a reinterpretation of Melissos's arguments to support Eleatic monism. Melissos re-presented Parmenides' arguments in Ionic prose, but he deviated from Parmenides' teachings. Parmenides claimed the real was a sphere, which suggests that the real was finite. Melissos claimed that the real was infinite.

> The real, he said, could only be limited by empty space, and there is no empty space.[1]

Melissos also presented a reductio argument against pluralism.

> If there were many things, they would have to be of the same description as I say the One is.[2]

While Parmenides had earlier advocated the spherical nature of the one, it was Melissos's assertion that there was no empty space that suggests the next development. Combining Parmenides sphere with the denial of both Melissos's assertions, that the real is infinite and that there is no empty space, yields a spherical non-infinite "real" in existing empty space. The denial of monism multiplies these non-infinite reals and produces atoms. That task fell to Leukippos.

Leukippos (450-420 bc)

> It is certain that Aristotle and Theophrastos both regarded [Leukippos] as the real author of the atomic theory.[3]

Leukippos modified Melissos's statement into the proposition that there are many things and that they are all spherical as Parmenides had said the one is, but not infinite. He also denied the non-existence of empty space. Creating empty space provides a "place" to put these many little "reals". This development can be seen as distinguishing between substance and existence. Prior to this distinction, substance was that which existed and non-substance or void was that which did not exist. When the two notions are distinguished, then it is possible to have non-substance that exists (void).

Prior to Leukippos the strong association of being with anything that could be thought of or said constituted a denial of a void in nature.[4] Whatever is thought of is, and the world is full of substance. Nothing could not exist and could not be thought. Arguments fairly raged about whether that substance was one or many, and if so, how many. But it was Leukippos who distinguished between non-existence and empty space.

> Leukippos supposed himself to have discovered a theory which would avoid this consequence [the impossibility of motion and multiplicity]. Leukippos was the first philosopher to affirm, with a full consciousness of what he was doing, the existence of empty space. The Pythagorean void had been more or less identified with 'air', but the void of Leukippos was really a vacuum.[5]

Taking this new distinction literally poses problems for Aristotle later when he struggles with point and place. (See page 120. below.)

Democritus (460-370 bc)

This distinction is also more clearly made when Democritus clarifies and expands Leukippos's theory.

> Leucippus had been content to speak of it, as did the Eleatics who denied that it existed at all, as the 'not-real' or 'non-existent' (μὴ ὄν): according to Aristotle, Democritus, taking advantage of the distinction between the two Greek negatives, called it the 'unreal' (οὐκ ὄν) or the 'nothing' (οὐδέν). He was in this way able to distinguish the void whose existence he affirmed as stoutly as Leucippus from absolute non-existence (τὸ μὴ ὄν), and to dispose of his opponents' objections by phraseology as well as argument. . . . 'space' was not 'the real'(ὄν), not body, neither was it the

> 'not-real' (μὴ ὄν), that which does not exist at all, but only 'unreal' (οὐκ ὄν). The distinction is a strong reinforcement of what Leucippus meant.[6]

> No one, even in modern times, has given a more classic expression to atomism . . . the only differences allowed to the elements are strictly geometrical, plus the motion in space necessary to alter their positions. For Democritus therefore two principles explain everything: atoms and empty space. . . . In the first place, each atom is indivisible. The word *atom* itself means indivisible; it was for that reason that Democritus invented the term *atom* and applied it to his elements.[7]

A number of things come together here. Matter and space are distinguished from existence and non-existence. It is denied that matter has both infinite extent and infinite divisibility. But plurality is preserved by limiting its scope. This actually foreshadows the development of recursion in the twentieth century. It also disposes of the problem of infinite regress implicit in the notion of infinite divisibility. We also have each of the types of matter previously conceived preserved as individual types of atoms. This polished atomic theory holds together quite nicely, although its incompatibility with geometry will readily become apparent. We now turn to Aristotle's struggles with this and his rejection of the atomic theory.

Aristotle

Infinite divisibility had already long been a topic of philosophical discussion before The Philosopher came on the scene. Aristotle, in his characteristic way, summarized selected arguments about it in his On Generation and Corruption[8] and, to a lesser extent, in The Physics.[9] He accompanied these summaries with critical analyses which include his reasons for rejecting the views of the atomists. Aristotle is not putting forward a positive model of infinite divisibility; he is presenting the horns of the atomists' dilemma and rejecting both horns.

The atomists say that there must be atoms, because things being infinitely divisible leads to the absurd conclusion that there is nothing left to reassemble. Aristotle argues that although things are infinitely divisible, they are not divisible everywhere. He thus rejects a premiss of the atomists' necessary to conclude that nothing is left to reassemble. He does not go further and present a positive account of infinite divisibility.

Here is the abstract form of the atomists' and Aristotle's arguments.

The Atomists:	Aristotle:
Either A OR B.	Either A OR (B′ AND NOT B).
∴ IF NOT B THEN A.	∴ IF (B′ AND NOT B) THEN NOT A.
IF B THEN C.	IF X THEN (B′ AND NOT B).
NOT C.	X.
∴ Conclude NOT B.	∴ Conclude (B′ AND NOT B).
∴ Conclude A.	∴ Conclude NOT A.

A - Reality is made up of atoms.
B - Reality is everywhere divisible.
B′ - Reality is anywhere divisible.
C - There is nothing left to reassemble.
X - Point is not contiguous to another point.

There are flaws in both arguments. In what follows, I shall examine Aristotle's critical analyses and supporting definitions from various perspectives. I shall contrast his writings with our current understanding of mathematical infinity, the real-number line, and model-theoretic semantics. My aim is to show that the use of these perspectives allows us to identify problems with his analyses that are otherwise difficult to uncover and to provide some account for one of his more opaque passages. I shall begin by examining his supplementary definitions.

Quantity

Only divisible quantities may be considered as candidates for infinite divisibility, and Aristotle defines what a quantity is in The Categories.[10] In 4b20 he divides quantity into discrete and continuous. According to Aristotle, number and language are discrete, and lines, surfaces, bodies, time, and place are all continuous (4b22). Aristotle distinguishes between discrete and continuous in the way their parts interact. The parts of discrete quantities do not have common boundaries; the parts of continuous quantities do. Aristotle is also explicit in stating that nothing else is a quantity:

> Only these we have mentioned are called quantities strictly; all the others derivatively; for it is to these we look when we call the others quantities. (5a38)[11]

Aristotle, in The Metaphysics[12], strongly suggests that quantities, and only quantities, are divisible.

> 'Quantum' means that which is divisible into two or more constituent parts of which each is by nature a 'one' and a 'this'. A quantum is a plurality if it is numerable, a magnitude if it is measurable. 'Plurality' means that which is divisible potentially into non-continuous parts, 'magnitude' that which is divisible into continuous parts; (1020a7) . . . for these also are called quanta of a sort and continuous because the things of which these are attributes are divisible. (1020a30)

The foregoing illustrates how Aristotle has identified what are quantities and hence what are divisible. A quantity is something that is divisible; on the other hand, anything that is divisible is a quantity. As a result, the phrase 'divisible quantity' is redundant; however, it will often be useful to keep this redundancy in mind. Aristotle has, in effect, established that infinite divisibility can be analyzed only as "infinitely divisible quantity".

The Infinite

The nature of infinite divisibility depends upon the nature of the infinite. To examine the role the infinite plays in infinite divisibility, we must turn to Aristotle's discussion of the nature of the infinite in The Physics. In 202b30 - 203b14, he surveys the opinions of his predecessors concerning the nature of the infinite and includes in the survey opinions of those who affirm infinite divisibility as well as opinions of atomists who deny it.

Some of Aristotle's predecessors held the view that infinity was some real thing itself, a view which Aristotle objects to. In spite of the reification of infinity suggested by his own use of the definite article in his discussions of "the infinite", Aristotle essentially settles on a definition of 'infinite' which rejects its being a proper subject. "Infinite", in his view, is predicated of other things -- magnitude and number in particular as well as time and motion. Aristotle said:

> Some of these do not treat infinity as an attribute of something else but make the infinite itself a substance; but of these the Pythagoreans treat it as present in sensible things, and also describe what is outside the heavens as infinite, while Plato recognizes nothing outside the heavens, but makes the infinite a constituent both of sensible things and of ideas. (203a4)

Aristotle rejects these views in favor of a more process-oriented view, one which spans the distinction between atomism and its opposition. He cites supporters of this view from both the atomist camp and their opposition, those who favor infinite divisibility:

> Those who make them infinite in number, as Anaxagoras and Democritus do, describe the infinite as continuous by contact. (203a20)

Aristotle supports the rejection of a reified infinity by arguing that the infinite is a principle.

> [The infinite's] being ungenerated and imperishable points to its being a principle; for there is a limit to all generation and destruction. This is why the infinite has no beginning but is itself thought to be the beginning of all other things . . . . (203b7)

This argument can be interpreted as follows: If the infinite were a thing then it would be something that could come to be and cease to be. The infinite cannot come to be nor cease to be. Since it cannot come to be nor cease to be, it must not be a thing.

In 203b15-24, Aristotle offers the following as supporting the existence of the infinite.

> Belief in the infinite is derived from five sources: (1) from the infinity of time, (2) from the divisibility of magnitudes, (3) from the fact that the perpetuity of generation and destruction can

> be maintained only if there is an infinite source to draw upon, (4) from the fact that the limited is always limited by something else; but above all, the infinity of number, of magnitudes, and of what is outside the heavens is inferred from (5) the fact that there is no limit to our power of thinking of them; (203b15)

Aristotle's statements of these beliefs seem somewhat question-begging or circular, but they all hint at something which can be continued again and again (presumably without end). He clearly favors the perspective on infinity which treats infinite as an attribute of a process which cannot be gone through. He would certainly agree that the infinite is not a thing which can come to be and cease to be; infinite is that attribute which identifies a process as having no end. Aristotle explicitly gives four senses of 'infinite', three of which include the notion of a process which cannot be completed.

> We must first distinguish the senses of 'infinite': (1) That whose nature forbids its being traversed, (2) that which admits of incomplete or (3) difficult traversal, or (4) which, though of such a nature as to be traversable, yet does not admit of it. Again what is infinite is so in respect of addition, of division, or of both. (204a2)

Even though Aristotle's language treats "infinite" as an object, these three senses clearly emphasize the interminable nature of processes that are said to be infinite. That such a process is said to be infinite makes it a "thing" which 'the infinite' is predicated of. In his continuing discussion, Aristotle becomes more explicit in this regard by stating:

> For infinity is an attribute of number and magnitude, and an attribute of an attribute is even less capable of independent existence than an attribute. (204a15)

Of course, attributes are predicated of other things. (Infinite is an attribute predicated of divisibility and divisibility is an attribute predicated of magnitude.) Once the infinite is predicated of processes, and limited to the process of addition and division at that, the subjects of the processes come under examination.

Aristotle claims that a thing is infinite only by addition or by division. This is more precisely stated by saying that a thing is infinite only by *the process of* addition or division. Extracting the essence of this in regard to the infinite yields "the process is infinite". "Infinite" is then predicated of a subject. The only subjects he deems appropriate are magnitude, number, time, and motion. In regard to addition, Aristotle concludes, "Clearly then there is no actually existent infinite body" (206a7), a conclusion with which modern science agrees.[13] He goes on to apply divisibility to extension.

> Spatial extension is not infinite in actuality, but is so (a) by division (the belief in indivisible lines is easily refuted); (206a17)

So, the infinite is not actually predicated of magnitude per se. It is the divisibility of magnitude -- the process of dividing -- that infinite is actually predicated of.

Atomism

Aristotle's discussion of infinite divisibility in The Physics is intended mostly to elucidate "the infinite" rather than to address the topic of infinite divisibility itself. The parenthetical insert at 206a17 is not justified. It appears to be question-begging. Suppose the atomists were right. First consider the simple case of dividing a line by the process of bisection. If each line were always divisible exactly in half, it would have had to have had an even number of points. Not only would the number of points have to be even, it would have to be an exact power of two -- 2, 4, 8, 16, 32, ... The consequence would be for lines of four and eight points to exist but not for lines of six points. Bisecting a line of length six yields two lines of length three -- a length not capable of being bisected. This is clearly absurd (except, perhaps, to the Pythagoreans).[14] The alternative would be for a line to have had an unending supply of points. But having an unending supply of points is just having an unending supply of places where it is divisible. Therefore, to presume that a line is divisible exactly in half is to presume that it is infinitely divisible. Rather than refuting atomism, this begs the question by assuming infinite divisibility.

Let's examine a bit further the consequences of the atomists' position. Under their presumption, lines will be composed of a finite number of (indivisible and extended) points. To presume that a line may be divisible into two (not necessarily equal) parts is to presume that the line is at least as long as the magnitude of two (adjacent) points. Since a line must consist of at least two points by definition, a line is always divisible into two parts. But the two parts may be single (extended and indivisible) points and hence not proper lines. We can call such a part a *degenerate* or *improper* "line". These improper lines are not themselves divisible; hence there is a limit to divisibility (under the presumed atomic structure). Consequently, that there are no indivisible lines, per se, does not necessitate that lines are infinitely divisible. The atomists' position is not as fragile as Aristotle would have us believe; he believes lines are continuous, and his belief has infected his reasoning. His parenthetical remark at 206a17 is gratuitous. It reminds one of the remark Fermat scribbled in the margin of a book -- the remark about the existence of an alleged "elegant and simple" proof of his famous last theorem (which was proven in 1993, but the proof was not trivial).

Infinite Divisibility and Matter

In The Physics, Book III, Chapter 7, Aristotle asserts a reciprocal relationship between number and magnitude. Number is potentially infinite by addition just as magnitude is potentially infinite by division. Number has a minimum unit while magnitude has an (unspecified) maximum. He concludes:

(1) Magnitude is infinitely divisible.
(2) Number is infinitely addable.

Peano's successor axiom makes (2) explicit with regard to number. "Every number has a successor" captures the notion that number has the nature of being, as Aristotle would say, "infinite by addition"; addition is a never-ending process. Consequently, I cannot argue with (2).

But, it seems to me, (1) can be questioned. Hume and Berkeley both argue against it.

Infinite Divisibility

I have approached Aristotle's treatment of "infinite divisibility" by looking first at the logically prior treatments of (divisible) "quantity" in The Categories and "the infinite" in The Physics. Aristotle deals specifically with "infinite divisibility" in On Generation and Corruption. In chapter two of that work he attempts to clarify what 'infinite divisibility' means. He needs to clarify the meaning of 'infinite divisibility', because, as he sees it, several important notions (coming to be, alteration, growth, and undergoing the contrary of these) all depend upon how infinite divisibility is characterized.

> Basic to all this is the question whether the things there are come to be and alter and grow and undergo the contrary of these things because the primary existences are things which have size, and are indivisible, or whether nothing which has size is indivisible; this makes a great deal of difference. (315b24)

On a first reading of this chapter, one might think that Aristotle had two models which were not compatible and that their incompatibility could be accounted for by Georg Cantor's account of transfinite numbers. That is, Aristotle could have devised a coherent characterization of "infinite divisibility" had he known of the different orders of infinity as characterized by Cantor.[15] Aristotle was getting apparently contradictory views by tacitly assuming (falsely) that there is only one kind of infinity.

As a result of Cantor's work, we differentiate among infinities of different cardinality (size). The first such division is between the size of the natural numbers and the real numbers, and lines are usually modeled by real numbers. Some of Aristotle's descriptions conform to a cardinality characteristic of the natural numbers and some to a cardinality characteristic of the real-number line.

On a more careful analysis of Aristotle's passages on infinite divisibility, and in consideration of the types of language he uses in describing his analysis, a more subtle reading is possible. The need for different infinities may be too strong. A lesser difference may be sufficient to account for the problem. The difference here is the distinction between so-called "discrete" sets and "dense" sets.

In this particular use of 'discrete', the members of such a set can be placed "next" to each other and be counted. Two elements can be said to be next to each other, successive, or adjacent, if there are no other elements between them. Aristotle's corresponding definitions can be found in The Physics. He defines 'in succession' and 'between' in book 5, chapter 3, in the context of a discussion of motion. For our purposes here, only certain aspects of these definitions are necessary.

'Between' involves at least three terms; (226b26)

> That is 'in succession' which, being after . . . has nothing of the *same* kind between it and what it is in succession to. (226b34)

The natural numbers, with the successor function, is the primary exemplar of such a set and represents the discrete model. In the case of any set of macroscopic objects which are stacked or lined up and can be counted, the "successor" or "next" relation can be shown of any given one of such objects (except the last one in a finite set).

Sets which are "dense" have the property that between any two members of the set another can be found. The real numbers provide the most common example. If you have two real numbers x and y, then the number (x+y)/2 is between x and y.

One's first reaction to Aristotle's problem with infinite divisibility might be to consider the difference between the natural numbers with their successor property and the real numbers with their dense property as providing a way of explaining Aristotle's difficulty. These two sets represent different orders of infinity, so one might be tempted to think the different orders of infinity are required to extricate Aristotle from his difficulties.

In the problem of infinite divisibility, it is the dense property of the real numbers which suggests a difference sufficient to distinguish between the two models. It is not the greater cardinality of the real numbers which is significant, merely their denseness. Another dense set is the set of rational numbers. If you have rational numbers a/b and c/d, then the number (a/b+c/d)/2, which is just the rational number (ad+bc)/2bd, is between a/b and c/d. Just as with the real numbers, between any two rational numbers is a third. But rational numbers have the same cardinality as the natural numbers. The distinction between the dense and discrete property may be adequate to explain the difficulty with infinite divisibility.

One potential source of difficulty is that Aristotle views magnitude as a continuous quantity. "The 'continuous'", says Aristotle, "is a species of the contiguous; two things are continuous when the limits of each become identical and are held together." (227a7) He had previously proposed a definition for 'contiguous' in 227a6: "That is 'contiguous' which, being in succession, is also in contact."

The rational numbers do not form a continuous set. In fact, the set of rational numbers is discontinuous "everywhere" because there are irrational numbers arbitrarily close to any rational number. There are infinitely many irrational "holes" in the (rational) number line. Aristotle's definition of 'continuous' might seem to require a greater infinity than the countable. ('Countably infinite' means having the same cardinality as the natural numbers.) Even the real-number line seems to fail to satisfy his definition of 'continuous'. Two things are continuous if they are of the same kind, are contiguous, and touch. Certain sets of real numbers -- intervals -- can touch and, when touching, form a unity. But two (distinct) single real numbers cannot touch any more than two rational numbers can.

If we ignore the irrational "holes" in the number line, then sets of rational numbers behave just like real numbers as above. In other words, relative to the set of rational numbers, closed interval sets of rational numbers can form contiguous, touching, consecutive sets of the same kind. (They "touch" provided their terminus is a rational number and not an irrational one. For example, the closed interval ending with 2 touches the closed interval beginning with 2. But the open interval terminated by π does not touch the open interval beginning at π since π is irrational.) So, some closed interval sets of rational numbers are continuous qua the rationals. Half-open intervals would seem to qualify, but they do not touch, even though there is nothing between them.

Potential and Actual

The potential/actual distinction applied to divisibility seems to correlate with these two models. That which is potentially divided forms contiguous parts with a common boundary wherever it might be actually divided. The actually infinitely divided would give an exploded view with adjacent parts "next" to one another (but not touching -- as in discrete sets). These views are irreconcilable in that the potentially divided retains its dense structure, while the actually divided does not. Now, we can find a mapping from the natural numbers to the rational numbers showing that there is still the "same number" of points. But the order required by "next" gets changed around.

In 316a16-25, as Aristotle summarizes the Atomists' argument, he is clearly using a dense model when he argues that no body or thing possessed of size is left, since, if anything left had size, it would also be divisible. Such a body has the same structure as denseness in that a point exists between any two distinct points.

In 316a26-33 he is discussing the characteristics of "sizeless" points. Since points have no size, when two of them are placed together they become coincident; that is, only one point remains. Moreover, an object composed of only two points, when divided, still retains the same overall size (two points of no size still adds up to no size). However, generalizing from combining any finite number of points to combining infinitely many points is not valid. But this is just what Aristotle does. He asserts that an object assembled from ("infinitely") many sizeless points cannot have size:

> So even if all the points are put together they will not produce size. (316a34)

One suspects that Aristotle doesn't have it all together in view of his previous statement:

> Similarly, if it is formed out of points it will not have quantity; for when the points were in contact and there was just one thing possessed of size and they were together, they did not make the whole the slightest bit larger; (316a28)

Aristotle is a bit loose here. Points cannot be in contact without being coincident. In his notes, Williams seems aware of the errors -- he adds quotes to 'in contact' and 'together' -- but he focuses on the issue of continuity and questions Aristotle's mastery of the argument involving the "sawdust model".[16] Joachim notes a grammatical shift of subject at the same point (or is it "place"?)[17]

It seems clear to me that Aristotle's mental model involves "exploding" a body of size into discretely separated sizeless parts (points) and then sequentially putting together points two at a time. He is left with only one point after each step. This process can continue for a countably infinite number of steps and still yield the same result: an object the size of one point -- "a set of measure zero".

Something is intuitively wrong when the reassembled parts do not make up the original whole. The flaw is in presuming that "exploding" the original yields discrete consecutive points, that is, gives the same result as a completed infinite division. Mentally explode something <u>slowly</u>; it stretches rather than breaks. The dense nature of a continuous object supplies "as many more points as necessary" to fill in any gaps where it might be [is potentially] divided. Mathematically, an interval of any size can be transformed into one of any other size, including between the finite and the infinite. I will return to this subject later; for now, it is sufficient to comment on the difference between the discrete "exploded view" and the dense "assembled view".

This difference corresponds to the distinction between counting and measuring, about which Ackrill, in his comments on The Categories, says:

> Aristotle does not stop to examine carefully the nature of counting and measuring, nor does he survey the different ways in which quantity or quantities may be spoken of; . . .[18]

If 'quantity' represents an abstraction subsuming both counted quantities and measured quantities and these are somewhat conflated, as Ackrill implicitly suggests, then Aristotle's apparent shift between a discrete and a dense model is understandable.

Infinity Times Zero

There is a problem with Aristotle's use of 'all' when he refers to putting "all" the points together. 'All', in this sense, stands for an undistinguished infinity. By adding all the points together, we have the sum over infinity of a zero-sized body. Summing the same thing many times amounts to multiplying by the number of times; for example, adding 4 a total of 10 times is the same as multiplying 4 times 10. The net result is that adding all the points has the structure of multiplying zero times infinity; that product is mathematically indeterminate.

If we coordinate the division process with the summing process, we can overcome this indeterminate result. Suppose we divide a body into N pieces, each of size L/N where L stands for the length or size of the object. (Remember, Aristotle's conception of the infinite is of a process which cannot be gone through. Dividing a magnitude in an attempt to achieve infinite divisibility by such a process is dividing it with an ever-increasing number of divisions and cannot actually be completed.) Something is infinitely divisible if N is getting larger and larger. Conversely, L/N is getting smaller and smaller, until finally (if there can be a finally), N reaches infinity and L/N reaches zero size. Of course, according to Aristotle, the infinite cannot be gone through; the limit cannot be reached by any direct method. Since the infinite cannot be gone through, any attempt to put things back together must be done with the incompletely divided fragments -- which are of size L/N.

By coordinating the summing with the division, we perform the multiplication of size L/N times the number of segments N and obtain a product of $(L/N)\cdot N$. Notice that this simplifies to L, and it no longer matters how big N is. So, if we take the limit as N approaches infinity of $(L/N)\cdot N$, we end up with L, the same size we started with. Aristotle and the atomists neglect the fact that infinity cannot be gone through when they presume something to be infinitely divided. Additionally, both the atomists and Aristotle implicitly presume that infinity times zero is zero. Of course there would be nothing reassembled if this were true. But infinity times zero is not zero; it is indeterminate. The atomists' premiss, that there would be nothing reassemblable, is not compelling. The reductio argument fails.

The Atomists' Argument Expanded

The atomists' argument is flawed in another way, as is Aristotle's presentation. Here is a greatly expanded version of the argument.

1. Every perceptible body is potentially divisible at every point.

2. It is impossible that a body is actually divisible at every point (simultaneously). (Premiss 2 is proven by reductio in conjunction with premiss 3.)

3. Nothing can come to be out of nothing or cease to be into nothing. (319a16-21)

2.1. It is possible that a perceptible body is actually divided at any point. (Assume the contrary of 2.)

2.2. If a body is actually divided at every point, then there will be nothing left. (Premiss 2.2 is itself proven by reductio.)

2.2.1. If there were something left, it could be further divided at some point, contradicting its having actually been divided at every point.

2.3. If nothing is left, then the body will have vanished into something incorporeal.

2.4. If a body vanishes into something incorporeal, then it ceases to be (something corporeal).

2.5. If something can cease to be something corporeal, then it can also come to be something corporeal (out of points or out of nothing at all).

Therefore, 2.1 leads to the contrary of 3 and absurdity; 2 is proven by reductio.

If a body is not actually (simultaneously) divisible at every point, then consider whether it is potentially (non-simultaneously) divisible at every point.

4.1. If a corporeal body is divided, it is divided into corporeal bodies. (A trivial case is division into a corporeal body and separate points. The corporeal body is not diminished by this form of division.)

4.2. Division into parts cannot yield a process which goes on to infinity because infinity cannot be gone through. Non-simultaneous division is a process of successive divisions which, by the nature of the infinite, cannot be completed. Any stopping point would yield, by 4.1, undivided corporeal bodies.

So, a body cannot be divisible everywhere because either the process could not be completed and something of size would be left, or nothing at all would be left.

The Atomists are happy with the first horn but must further reject the second horn. That rejection flows as follows:

5. If the process *were* to be carried out to infinity (or simultaneously), the parts would all be nothing at all or vanishingly small points.

6. If coming to be and ceasing to be are to take place by aggregation and segregation, then aggregation must be capable of adding to the size of what comes to be; conversely, segregation must diminish the size of what ceases to be.

7. Aggregation cannot proceed by the accumulation of nothing at all or points (vanishingly small pieces). Conversely, segregation cannot proceed by diminution from vanishingly small pieces or points.

8. Coming to be takes place by aggregation; ceasing to be takes place by segregation.

9. Aggregation must proceed by the addition of pieces of some determinate size. Conversely it is so with segregation.

Here the atomists' argument actually makes a wild leap of faith.

10. The process of aggregation itself proceeds by exactly the size of the limit of divisibility, and that is the minimum size (which just happens to be so small as to be invisible).

That aggregation must proceed by the addition of pieces of some determinate size begs the question if "determinate" is taken to mean "minimum". The argument is that aggregation must occur by the addition of something of size. Nothing is actually presented to rule out continuous accretion. If there were some non-question-begging way to rule out continuous accretion, we would have a nice, tight reductio. But it leaks.

Aristotle has it that aggregation occurs by "leaps and bounds", that is by the addition of large clumps of material at each "step". These clumps are themselves infinitely divisible but not divisible at every point. He rejects the atomists' solution, retains infinite divisibility, but rejects the notion that objects of size are everywhere divisible. (The real-number line satisfies this by having a countably infinite number of rational "division points" and many irrational "non-division points" but does so by including different orders of infinity.) Of course, Aristotle believes that time is infinitely divisible; there is no problem with half as big an increase occurring in half the time. Continuous accretion is the natural consequence of this model. However, aggregation by atoms goes along equally well with a limit to divisibility of both magnitude and duration. The nature of time (continuous or discrete) would seem to fit in the appropriate model. Even if duration were not infinitely divisible, aggregation could still occur by the addition of divisible clumps at each interval.

Aristotle, however, does not proceed along this line; he takes another tack.

"Contact"

In 316b6 Aristotle introduces a statement from another level of analysis. He states:

> And any one contact always involves two things, since there has to be something else besides the contact or division point.

Aristotle asserts that any one contact requires two things, in the sense of distinct things. This requirement seems to be based upon a syntactic-level notion for the word 'contact'. Contact requires two distinct objects touching or "contacting" at a single point. Remaining distinct while still sharing a point (of contact), in truth, requires a minimum of _3_ points. Since there is no reason to prefer one object over another, each must provide a point distinct from the point of contact. Otherwise, one object would not be distinct from the point of contact, which is part of the other object as well. We would, in such a case, have one object with a point of contact and no other object. Aristotle's argument is a little weak here, since he asserts only the need for 2 points instead of 3.

His mistake is understandable in the light of the tension between incompatible models: discrete quantities do not have a point of contact; continuous quantities "join" at a point (of contact). Since two distinct discrete quantities do not share a point of contact, no third "point of contact" is required.

This kind of characterization of contact has as a prime paradigm such things as stacks of coins or columns of bricks. Macroscopic objects all have at least two distinct points (and all points between) so can remain distinct apart from a shared "point of contact". Further, the type of such a set of objects is discrete. One can use the point of contact along with the other two distinct points (one for each of the two objects) to evoke the adjacent, or next, relationship. Starting with the point not of contact in object "A" (and we are guaranteed that at least one such point exists), move first to the point of contact and then to the point not of contact in object "B" (and we are guaranteed that at least one of these points exists also). By this method we move from one object to "the next" (a notion Aristotle uses). In doing this, we have ignored any points of objects "A" & "B" except for one point not of contact from each object and the point of contact.

<u>Anywhere is not Everywhere</u>

For parts to form a continuous whole, there must be a point of contact between one part and "the next". If something were divisible everywhere, the parts would be mere points -- which, according to Aristotle, cannot be recombined into anything of size. Aristotle must find a way to disallow divisibility everywhere while still permitting infinite divisibility. Lear, in "Aristotelian Infinity", notes:

> Aristotle offers a paradigmatically Aristotelian solution. He distinguishes two senses in which a line may be said to be divisible 'through and through' (317a3ff). A length is divisible through and through in the sense that it could be divided *anywhere* along its length. But it is not divisible through and through in the sense that it could (even potentially) be divided *everywhere* along the length. One can thus actualize *any* point but one cannot actualize *every* point; for any process of division, there must be divisions which could have been made which in fact were not made.[19]

Lear, who is more interested in infinity than infinite divisibility, takes the potential/actual distinction at face value and does not elaborate further.

In 317a3 Aristotle states:

> Since no point is contiguous to another point, there is one sense in which divisibility at every point belongs to things of size and another in which it does not.

The distinction between these senses will be clear when we examine his argument in the next few lines. It follows from an apparent distinction between his usages of "anywhere" and "everywhere". He goes on to state:

> When this is asserted, it is thought that there is a point both anywhere and everywhere, so that the magnitude has necessarily to be divided up into nothing; . . . (317a5)

Here Aristotle begins to hint at his argument by implying a distinction between (divisibility) "anywhere" (hopēoun) and "everywhere" (pantē) and by using the conjunction "both . . . and" (kai . . . kai) to join them. The rest of the sentence,

> for because there is a point everywhere, it is formed either out of contacts or out of points (317a6),

introduces his distinction between places where the line is divisible (contacts) and places where it is not (points).

Aristotle continues with his analysis and refutation of the atomists' argument. He is quite terse in his dealings with it, and the brevity of his rebuttal leaves much to be desired. Williams comments:

> The above paraphrase of 317a2-12 is the nearest I can come to making sense of this baffling passage. A large part of it, 317a8-12, is so resistant to my attempts to understand it that I have contented myself with a literal translation which I have placed between obeli to indicate that no claim is made to have found a sure way of making sense of the Greek. Other commentators and translators seem to have fared no better, and I can hope to surpass them only in frankness.[20]

Better sense, I think, can be made of Aristotle's discussion by fleshing out certain contextual presumptions.

> When this is asserted, it is thought that there is a point [where something with magnitude is] both [divisible] anywhere and [divisible] everywhere, ... (317a5)

Aristotle is leading to the conclusion that something with magnitude is divisible anywhere but is not divisible everywhere, and since it is not divisible everywhere the atomists' argument is defeated. This distinction can be interpreted as corresponding to the distinction between being potentially divided and being actually divided. Aristotle seems actually to agree with the atomists, though he potentially does not. Although he agrees that there is a sense in which something with magnitude is not everywhere divisible, he denies that it follows from this that magnitude is not infinitely divisible.

Aristotle seems to connect 'anywhere' with "potentially divisible" in the following passage:

> In one sense there is a point everywhere, because there is one anywhere and all are like each one; . . . (317a8)

Here, the distinction between 'anywhere' and 'everywhere' is used in an entailment manner: "If there is a point 'anywhere' then there is a point 'everywhere'", with the reason given that "all are like each one". By this Aristotle means that there is no reason to suppose that one point is any different from any other. Potentially, there is a point anywhere.

He goes on with his crucial "point" that "but there is no more than one." By this he means that there is no more than one point anywhere (at each position). His argument to support this immediately follows: "since they are not consecutive, . . .". (317a9)

One might expect 'contiguous' in place of 'consecutive' above. 'Contiguous' would seem to fit more easily in the reading that follows. His use of 'consecutive' at this point seems more in keeping with the undivided, or potentially divided, state under consideration. The undivided line is continuous, so any parts are contiguous to each other; in this state, the parts are not discretely separated by actual division. Consequently, there is no "next" part adjacent to any chosen one. So, two points (parts) are <u>not consecutive</u>. (But they are not contiguous either -- the "parts" of something continuous are normally contiguous, but points are an exception.) Aristotle concludes:

> so it is not the case that there is a point everywhere. (317a9)

<u>Point and "Place"</u>

Sense can be made of the argument at this point if we interpret Aristotle to be thinking of both point and place but ambiguously using the term 'point' for both. We need to recall that, for Aristotle, a point must be actualized in some way. He would say that the place of a potential point does not hold an actual point. We can distinguish between a place where there is an actualized point and a place where there is no (actualized) point (there would be a *potential* point in such a place). By the symmetry of arbitrariness, any place could have a point. Once such a place is chosen, then the contiguous places have no points. To show how this works, here is an expanded interpretation of the argument at (317a9)

> If there were points everywhere, then there would be contiguous points at every point. Contiguous points must touch. Since points take up no space at all, touching points would be in the same place. This would necessitate there being two contiguous points at every place. But there can't be two points at the same place. Consequently, points cannot be contiguous. Since points cannot be contiguous, the "place" of the potential contiguous point must be empty. So, there is a sense in which there is not a point everywhere.

This argument entails a presumption that "place" is both contiguous and consecutive. Every place has both a contiguous place and a consecutive place. So, since points can't be contiguous, there must be places where points aren't, namely the contiguous places.

This is the sense in which there isn't a point everywhere, while the sense in which there is a point anywhere is its potentially having been chosen as the starting "point". I think this establishes that Aristotle differs in his interpretation of (divisible) 'anywhere' and 'everywhere'. He has concluded that even though there is a point "anywhere" there is <u>not</u> a point "everywhere". There is a loose association between the term 'anywhere' and the potentially chosen starting point and the term ’everywhere’ and the contiguous places which might hold points -- loose, but not consistent.

There is a kind of logic in his argument which is illustrated by the following example. Suppose we want to talk about "the Universe" and intend by that term to include all things, including all space and all time. Then, suppose someone wants to use the term ’edge’, or ’boundary’, in talking about the universe as we have defined it. By ’edge’, or ’boundary’, we usually mean a distinction or division between two things, one of which belongs to one side and the other of which belongs to the other side of the distinction.

"Edge of the universe" presupposes that there is something which is in the universe and, on the other side of the edge, something which is not in the universe. But we defined ’universe’ as including all things, so we can conclude that there is no edge to the universe. That is, we cannot consistently use the term ’edge’, or ’boundary’, with its usual meaning, if we hold the meaning of ’universe’, as we have defined it, to include all things. It requires careful analysis to notice that simply using the term ’edge’, or ’boundary’, with ’universe’ introduces a contradiction and yields a (verbal) structure which is no longer consistent.

Aristotle seems to be displaying a similar type of argument, but one which is more subtle, in creating a distinction between the intended usages of the terms 'everywhere' and 'anywhere'. He allows for two senses of 'everywhere', in one of which there is a point (and hence divisibility there) and in the other of which there is not a point (and hence no divisibility there). The argument is confused because he does not create a separate term for each usage and, moreover, does not seem to be consistent in his usages of 'everywhere' and 'anywhere' in regard to the distinguished senses.

In the sentence that follows, Aristotle is referring to the "place" which is contiguous to any point:

> For if it is divisible at the middle it will also be divisible at a contiguous point. (317a10)

This appears to introduce a terse reductio supporting his previous sentence.

1. Suppose we have two points (anywhere) at which magnitude is divisible. (He seems to be presuming that he is considering two "consecutive" divisible points.)

2. Suppose it is divisible at the middle (between these two points).

2.1 Then it would also be divisible at a contiguous point. (Either could have been actualized.)

2.2 But it cannot be divisible at both the middle and a contiguous point because there is no contiguous point.

> "For position is not contiguous to position or point to point." (317a12)

(Actualizing one precludes actualizing the other, but by symmetry either both must be actualizable or neither can be. And we have shown that one cannot be, so neither can be actualized.)

3. Therefore, by reductio, it could not have been divisible at the middle.

The presumption that any place has both a contiguous place and a consecutive place allows one to conclude that there is a "point" contiguous to the middle point. But points cannot be contiguous (without being coincident).

This presumption allows us to confuse the notions of discrete sets and dense sets, notions which are otherwise incompatible. With our visual model of the real line we think of a continuous set as something in which no amount of "stretching" will create breaks or "holes". Of course, with the rational numbers, which could be thought of as having a "granular" quality, stretching does not separate the grains any more than before stretching. There are always more points between any two points, no matter how close they were to start with.

The presumption that any place has both a consecutive place and a contiguous place cannot be modeled, because it is inconsistent in the context of Aristotle's definitions of 'contiguous' and 'consecutive'. As such, it represents a "syntactic-level" constraint or connection between the two incompatible models.

Aristotle abandons his visual model in favor of this syntactic-level argument when he nonchalantly states "since they are not consecutive". (317a9) He thereby justifies his statement that there is no more than one point anywhere. What he seems to be doing is shifting levels of argument from semantic considerations of the visual model of the real line to more purely syntactic-level constraints. In the semantic-level analysis, which is driven by the visual model, statements are judged by their agreement with the model -- in this case the visual image of the real line. By appealing to the assertion that the points are not contiguous (presumably after division), he allows a shift to the discrete model which provides the appearance of solving the problem. By not consistently sticking to semantic-level considerations, or conversely to syntactic-level constraints, he permits a kind of inconsistency to invade his argument. This inconsistency results directly from the notion that any place has both a contiguous place and a consecutive place.

Aristotle notes that, once an object is divided (into points), no point has a contiguous neighbor; allowing that leads to a contradiction of the hypothesis that the object was divided at every point. He denies that points are contiguous but implicitly assumes that there is a next point, distinct from the "one anywhere", and that the intervening distance is not divisible. He concludes that there are no points in this interval.

His argument here is based upon a view of the divided object represented by an exploded set of separated points, presumably in an array similar to what we see when we look at a newsprint photo with a magnifying glass. This kind of view represents a discrete set of points, while the set of points in an undivided region of space (line segment, or disk in the plane) is a dense set. Aristotle has unwittingly converted from dense to discrete in the middle of his argument.

The remainder of his sentence, "and this is division or composition." (317a12) simply states that he has dealt with the problem of infinite divisibility and, its converse, composition.

In 317a15-18, Aristotle confirms this interpretation when he states:

> nor in such a way that division can occur everywhere (for this is what would happen if point were contiguous to point) . . . .

In 317a18, he states:

> but into smaller and yet smaller parts, and aggregation out of smaller <into greater>.

Here, he recalls his purpose in examining infinite divisibility, namely to shape the concepts of aggregation and segregation, which will relate to alteration and growth.

Aristotle does not carry the smaller and smaller to its logical limit because he has just argued that there is a sense in which divisibility at every point is not applicable to objects of size. The process of dividing something into smaller and smaller parts seems a simple concept to grasp and, aside from practical considerations, easy enough to carry out, but Aristotle has asserted that there are some places at which the division cannot occur -- those places where there isn't a point. He does not relate the sense in which "there is not a point everywhere" (317a9) to the process of consecutive division.

I would like to have the consequence that there is not a point everywhere directly related to the process, so I could have some model or definition of how the process misses these points. The consequence itself doesn't seem likely. It could be argued that any process which yields any potential division point, finite or otherwise, will always select the point which is "anywhere", making other points the "not-everywhere" ones. In other words, it doesn't matter which points you choose; you can't choose any of the ones at which the thing is not divisible. Non-divisible points are forever inaccessible. Aristotle would say that that's because there aren't any "points" at these places.

Aristotle does not connect the two distinct adjacency criteria in the notion of "place", as instantiated in the two incompatible models (discrete and continuous), and consequently fails to notice the resulting contradiction.

Since his purpose was not to clarify infinite divisibility itself but only to clarify it in relation to its use in accounting for coming to be, alteration, growth, and undergoing the contrary of these, he stopped short of unearthing the contradiction.

Many discussions have attempted to reconcile the apparent contradiction by use of the potential/actual distinction. There is a point everywhere in the sense that a magnitude is potentially divided at the point. On the other hand, the uncompletable nature of (the process of) infinity is such that any magnitude subject to successive division is never fully divided. So, there are places at which there is no point (where it is *actually* divided). This might seem to be a promising way to remove the contradiction; yet David Bostock has extended Zeno's arguments to show that the potential/actual distinction does not, in fact, work.[21] Aristotle uses the distinction to dispute Zeno's argument -- he argues that Achilles does not actualize an infinity of points on his way to the tortoise. (263a23) Now, one way for a point to be actualized is for something to stop at it (262a21), and, in order for something to reverse direction, it must stop at, and thus actualize, the point of reversal. Bostock extends Zeno's argument to a bouncing ball in a way that actualizes an infinity of points.

> But [Aristotle] was able to deny the (actual) existence of these points only by denying that they had been actualized, for Achilles was not supposed to have done anything at or to the points as he passed them. However this reply is surely not available to the revised version of Zeno's problem that I have just put forward, for the ball's motion is certainly divided into infinitely many parts by the infinitely many points which mark the top of each bounce, and these must surely be admitted to have been actualized in the course of bouncing. I conclude, then, that Aristotle does not after all have the right solution to Zeno's problem.[22]

Aristotle himself asserts that points are not contiguous. Any object made up of points cannot be divisible at "non-points" "between" non-contiguous points. Unfortunately, between any two points there is another, to any desired degree of precision, using our usual order relation. So, between any non-contiguous points there are other points at which the object is potentially divided. We now know that the axiom of choice is equivalent to the axiom "Every set can be well ordered."[23] (A set is well ordered if every subset has a least element.) This axiom makes no statement about the cardinality of a set or the order under which it exhibits denseness. When it comes to the real number line, we don't know what the well-order relation is, but when so ordered, the points, separated by that order relation, would be consecutive without intervening points; but this would not be by the usual order relation, '<' (under which continuous objects are dense).

Summary of Aristotle's Views

A tension pervades Aristotle's thoughts on the infinite in general and infinite divisibility in particular. One form of this tension is a conflict between the dense structure of the potentially divided or undivided model on the one hand and the discrete structure of the actually divided or exploded model on the other hand. The basic incompatibility of these two views is not discovered by Aristotle. He appears to shift back and forth between the two models using predominately syntactic-level arguments. This shift may be facilitated by his holding a view of "place" which is inconsistent with his view of "points". There are both contiguous and consecutive places, while there can't be both contiguous and consecutive points. Since there is no logical way to tell places from points aside from the definition that points go into place, the view is inconsistent at worst, ambivalent at best.

Aristotle, and his predecessors, also had an (understandably) immature understanding of infinity; he believed that, essentially, zero times infinity is zero -- whereas we now know the product to be indeterminate. Further, neither Aristotle nor his predecessors were aware that infinity could be differentiated into different "sizes" -- a distinction which could go a long way toward resolving difficulties with infinite divisibility.

In chapter VI I show that a careful analysis of the positions of atomism (page 167) and infinite divisibility (page 158) reveals that both points of view are internally self-consistent. Both perspectives coexist in a manner comparable to the wave/particle duality of modern physics. Many of Aristotle's insights are still current, but he neither proved that magnitude is infinitely divisible nor did he refute the atomists.

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3. Burnet, p. 76.

4. David Gallop, Parmenides of Elea, (Toronto: University of Toronto Press, 1984), p. 8.

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6. Cyril Bailey, The Greek Atomists and Epicurus, (New York: Russell & Russell, 1964), pp. 118-9.

7. Gordon H. Clark, Thales to Dewey: A History of Philosophy, (Boston: The Riverside Press, 1957), p. 35.

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10. J. L. Ackrill, Aristotle's Categories and De Interpretatione, (Oxford: Clarendon Press, 1963).

11. Ackrill qualifies his use of 'quantity' in the translation. "Quantity: The Greek is a word that serves both as an interrogative and as an indefinite adjective (Latin *quantum*).", p. 77.

12. Aristotle, "Metaphysics", trans. W. D. Ross, in The Basic Works of Aristotle, ed. Richard McKeon (New York: Random House, 1941).

13. It is generally believed today that the universe is "finite and unbounded" with its closure remaining an open question. According to Heinz R. Pagels, Perfect Symmetry: The Search for the Beginning of Time, (New York: Bantam Books, 1986), p. 146,

> Today most scientists maintain that the universe evolved from a hot, dense gas of quantum particles

which subsequently expanded rapidly -- an explosion called the "hot big bang".

If the universe is "closed", the expansion will eventually stop and reverse -- yielding a finite universe. If the universe is "open", the expansion will continue, as Aristotle would say, without being gone through. The resulting universe will be finite at any moment in time, although it continues to expand. It would be at most "potentially" infinite.

14. Ross, p. 542.

15. Georg Cantor, Contribution to the founding of the Theory of Transfinite Numbers, trans. Philip E. B. Jourdain, (n.p., England: Open Court Publishing Company, 1915; reprint ed., New York: Dover Publications, 1955).

16. Williams, pp. 69-70.

17. Harold H. Joachim, Aristotle on Coming-to-be & Passing-away, (Oxford, England: The Clarendon Press, 1922), p. 79.

18. Ackrill, p. 91.

19. Jonathan Lear, "Aristotelian Infinity", Proceedings of the Aristotelian Society 80, (1979/80): 199-200.

20. Williams, p. 74.

21. David Bostock, "Aristotle, Zeno and the Potentially Infinite", Proceedings of the Aristotelian Society 73, (1972-3): 37-51.

22. Bostock, p. 46.

23. Elliott Mendelson, Introduction to Mathematical Logic 2nd. ed., (New York: D. Van Norstrand, 1979): 9.

CHAPTER V

ATOMISM AND DIVISIONISM AFTER ARISTOTLE

Epicurus (341-270 bc)

Epicurus's argument regarding atomism and divisionism is similar to Aristotle's, except that Epicurus arrives at the opposite conclusion. Furley distinguishes among three kinds of indivisibility. They are physical (atom), theoretical, and perceptual. It is Furley's contention that Epicurus's Letter to Herodotus begs the question regarding the atom.

> I claim that the passage to be discussed offers no argument at all for the existence of a physical minimum, but assumes it.[1]

But Furley then summarizes Epicurus's premisses and includes a quotation of Epicurus's reasoning.

> Nothing comes into being out of nothing or passes away into nothing, and the universe is a closed system -- it has no relations with anything outside it. The irreducible contents of the universe are bodies and space; everything else can be reduced to these. The bodies in question are "physically indivisible and unchangeable, if all things are not to be destroyed into non-being but are to remain durable in the dissolution of compounds -- solid by nature, unable to be dissolved anywhere or anyhow. It follows that the first principles must be physically indivisible bodies".[2]

According to Furley, the argument goes as follows:

> Real things cannot be destroyed into 'non-being'; but unless there were a limit to physical divisibility this is what would happen; there therefore must be a limit to physical divisibility.[3]

The argument expands not unlike Aristotle's presentation, but is much simpler because the distinction between place and point is not involved.

1. No real thing can pass-away into non-being (by division or otherwise).
2. If something is infinitely divisible then it can be divided into non-being.
3. If something can not be divided into non-being then it can not be infinitely divisible. (Contrapositive of 2.)
4. No real thing can be infinitely divisible. (by 1. & 3. using quantitative logic)

An examination of premiss 2 suggests a flaw. When something extended is divided, the parts are extended. When these extended parts are divided, their parts are extended. There is no limit to this process. Since having no limit to the process is what we mean by "infinitely divisible", we always have extended parts. As the number of times an extension is bisected increases, the limit of the size of the remaining extension is indeed zero. But at every stage in the process what is yet to be bisected has non-zero extension. At no stage will dividing a non-zero extension yield zero extension.

There is a special case: removing the end-point of a closed interval removes a "piece" with zero extension, but the remaining part still has the same extension as the original. But this operation can only be performed twice -- once for each end of the line segment. Cantor examined continuing the process by removing individual points in a line. This extended process can continue for countably many removals and still not diminish the extension of the original segment. (Such a set of points as was removed is known as a set of measure zero.)

If it be argued that the removed single points somehow pass away into non-being, the extension of the remaining parts is not diminished at all. Hence even allowing points to pass away into non-being does not cause the object under consideration to pass away into non-being.

In the former case, dividing an extended object into extended objects does not cause the object to pass away into non-being even if the division process is continued to infinity. A trivial proof by mathematical induction on the

N^{th} bisection shows that for all N, the parts are extended.[4] Consequently, it is an error to conclude that that which is infinitely divisible is divisible into non-being. We may conclude that Premiss 2 is false.

Now let us consider the case of a minimum theoretical quantity (idea or conception), as Furley translates Epicurus' "Letter to Herodotus".

> (A) Moreover one must not suppose that in the limited body there are infinitely numerous parts, even parts of any size you like.[5]
>
> (B1) Therefore we must not only do away with division into smaller and smaller parts to infinity, so that we may not make everything weak and in our conceptions of the totals be compelled to grind away things that exist and let them go to waste into the non-existent,[6]
>
> (B2) but also we must not suppose that in finite bodies you continue to infinity in passing on from one part to another, even if the parts get smaller and smaller.[7]

Furley interprets Epicurus's argument as meaning that the alternative to theoretical atomism is essentially infinite regress of thought, which is unacceptable.

> We must reject infinite divisibility, [Epicurus] says, for otherwise we should make everything weak -- that is to say, when we tried to get a firm mental grasp . . . on the atoms, we should find them crumbling away into nothingness. Every time we thought we had arrived at the irreducible minima, we should have to admit that even these minima are divisible. And so our search for the reality of the atoms would be endlessly frustrated.[8]

The weakness is motivation for rejecting infinite divisibility. The impossibility of completing an infinite sequence of contemplation of parts is grounds for rejecting infinite divisibility. Here's how I see the argument expanded.

a: We clearly comprehend a whole finite object.

b: To comprehend a whole object, we must comprehend its parts.

c: If its parts are infinite in number, then we cannot complete a sequential process of comprehending each part.

d: Therefore, we cannot comprehend its parts.

e: Therefore, we cannot comprehend the whole object.

According to Furley, both arguments are theoretical: One deals with what would be left after an infinite number of divisions; the other deals with how such a thing might be comprehended. He suggests these correspond directly to two of Zeno's arguments.

> (C1) For when someone once says that there are infinite parts in something, however small they may be, it is impossible to see how this can still be finite in size; for obviously the infinite parts must be of some size, and whatever size they may happen to be, the size <of the body> would be infinite.[9]

It is not obvious that the "some size" that the infinite parts must have does not have zero as a limit. For the argument to hold, "some size" must have a limit greater than zero. The process of bisection reduces the size by half and has a limit of zero. Although every bisection starts with something of "some size" it yields parts which still have "some size". And each bisection yields parts which have "some size", there is no non-zero limiting size. This argument is infected by question begging in assuming that there is a positive limit to "some size" (atomism).

Epicurus suggests that a theoretical or "cognitive" minimum can be conceived of as "next" to something similar, and that this sequential, one at a time, cognition transverses the finite body.

> (C2) And if the finite body has an extremity which is distinguishable, even though it cannot be thought of in isolation, it must be that one thinks of the similar part next to this and that thus as one proceeds onward step by step it is possible, according to this opponent, to arrive at infinity in thought.[10]

But by using the term 'next' he introduces the atomic perspective. Furley notices a difficulty but fails explicitly to note the question begging nature of the assumption implicit in the notion of "next".

> We are considering someone's suggestion that there are [an] infinite [number of] [parts] in a finite body. Starting from one edge of the body we imagine a minute part of it, 'the extremity', inconceivable in isolation from the body. If we think of the part next to this extremity, we must necessarily think of another distinct part similar to the extremity itself. But according to our imaginary opponent, there are in our finite body an *infinite* number of such parts. So if we proceed in thought from one such part to another, it must be possible, when we traverse the whole object, to reach infinity in our thinking, which is absurd.[11]
>
> It will be seen that this argument needs support. It is not yet clear why the extremity is a minute *part*, nor why we can only think of the part *next* to the extremity as being similar to it. This support is provided in the next sentence, by an analogy with the visual minimum.[12]

According to Furley, Epicurean theory often depends upon analogy with the perceptible for explaining the imperceptible.[13] The existence of a *perceptual* minimum is taken to support the existence of imperceptible minimum.

> (D1) We must observe that the minimum in sensation, too, is neither quite the same as that which allows progression from one part to another, nor wholly unlike it; it has a certain similarity to things which allow progression, but it has no distinction of parts.[14]
>
> (D2) When because of the closeness of the resemblance we think we *can* make distinctions in it -- one part to this side, one to that -- what confronts us must be equal.[15]
>
> (D3) And we study these parts in succession, beginning from the first, and not all within the same area nor as touching each other part to part, but, in their own proper nature, measuring out the sizes, more of them for a larger one, fewer for a smaller.[16]

(E) This analogy, we must believe, is followed by the minimum in the atom; for in its smallness, clearly, it differs from that which is perceptible, but it follows the same analogy. For we have already stated that the atom has magnitude, in virtue of its analogy with the things of this world, just projecting something small on a large scale.[17]

(F) Further, we must take these minimum partless limits as providing for larger and smaller things the standard of measurement of their lengths, being themselves the primary units, for our use in studying by means of thought these invisible bodies. For the similarity between them and changeable things is sufficient to establish so much;[18]

In the light of modern knowledge of vision systems, there is an element of question begging in the use of the perceptual analogy. The difficulty comes from the structure of the visual receptors in our eyes. The retina of each eye is comprised of an array of thousands of receptor cells, each with a finite size. The fact that there are two kinds of such cells is of no consequence. We know that these cells respond by triggering the discharge of an optical neuron. Aside from the fact that these cells are either discharging or are quiescent is the matter of their physical layout on the retina. Incoming light signals that activate a single receptor cell produce a minimum perceptual experience. It does not matter whether we choose to view light as corpuscular in nature or as a wave. It is simply not possible to have a visual or perceptual experience with less than one whole cell of the retina activated. And since there are a finite number of discrete cells, this amounts to built in atomism of the perceptual apparatus.

The geometry of the eye requires that light from an object stimulating a single receptor cell be within a minimum angle. The most dense concentration of cells occurs within the fovea. (Light from an object strikes the fovea when we look directly at the object). But when only one cell is activated, it is not possible to determine if the object has a relative size smaller than the minimum angle subtended by the cell. Figure 2 shows a diagram of the angle that can subtend one retina cell. Notice that the light from objects smaller than this angle can activate the cell, but because the cell response is simply on or off, there is no information in the perceptual system about how big the image of the object may have been on the cell itself. The perceptual response is simply that it sees the smallest possible activation. (Anything less would be no cells activated at all.)

The visual system has an atomic structure. Its built in bias is to respond in atomic terms. Consequently, using an analogy to argue from the perceptual to the actual brings the atomic structure of the visual system and imposes it via the analogy onto the actual. This is a subtle form of ques-

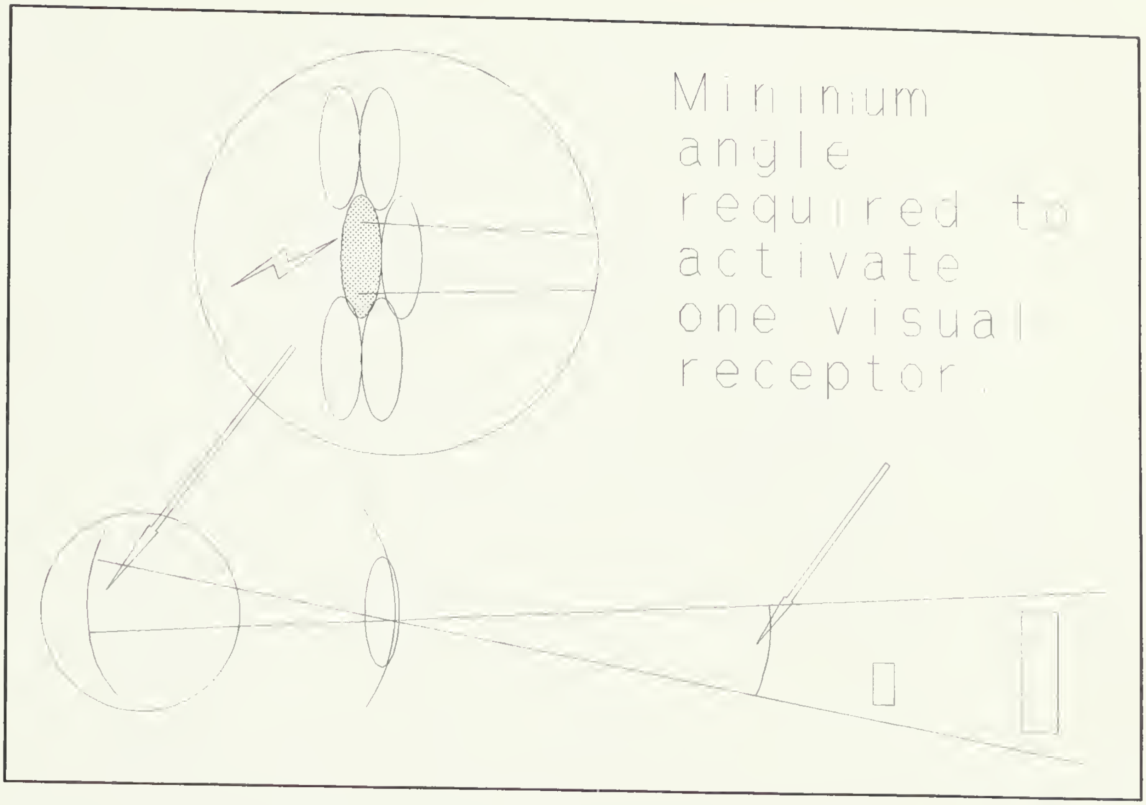

Figure 2. Visual apparatus involved in the minimum perceptual experience.

tion begging. "There are atoms because I see atoms." But I see atoms because my vision system has an atomic structure, and it shapes it's incoming information into its own atomic structure.

Chrysippus (280-207 bc)

Although Zeno of Citum (334-262 bc) founded the Stoic school, its view is largely known through Chrysippus. The stoics, we are told, rejected atomism. The rejection appears to have centered around problems with infinity. Today it is well known that both the natural numbers and the even numbers are infinite; the even numbers are a part of the natural numbers. There is no difference [in number] between them. It would be inappropriate to interpret 'no difference' in terms of subtraction -- infinity minus infinity is indeterminate. But it appears that Chrysippus may have been onto one of the paradoxes of infinity.

> Chrysippus, we are told, held "bodies" to be infinitely divisible, not in the sense that a body could be divided into an infinite number of parts, but in the sense that there was no limit to division. It followed from this, as he observed, that there was no sense in saying that the whole of any extended magnitude contained more parts than any one of its parts. "Man does not consist of more parts than his finger, nor the cosmos of more parts than a man; for division of bodies continues to infinity, and of infinities none is greater or smaller than others." This Stoic doctrine is a more precise and deliberate formulation of a principle first announced by Anaxagoras: "Of the small there is no smallest, but always a smaller, since what exists cannot cease to exist; also there is always a larger than the large . . . ." It is worth noting, too, that Chrysippus appears to have avoided saying that two infinities are *equal*; he said that no infinity is *greater or smaller* than another.[19]

In the light of this view, questions would arise concerning what the terms 'more', 'less', and 'same' mean in the context of the infinite. Obviously, lack of recognition that 'same' [size] in the context of infinity could mean different things, allows equivocation to creep into arguments about it. But Chrysippus's care in this matter seems not to have been followed by Lucretius.

<u>Lucretius (99/94-55/51 bc)</u>

Lucretius seems to be aware of what could both be described as a characteristic of and as a problem with infinity. One modern way to show that a set is infinite is to show that it can be placed in one-to-one correspondence with a proper subset of itself. For every integer there is a corresponding even integer. Multiply the integer by 2 to get the even integer; divide the even integer by 2 to get the integer. It is this very thing that causes Lucretius to reject infinite divisibility.

> The argument for the existence of *minimae partes* is worth a little attention, since it seems to be something not found in the *Letter to Herodotus*. The argument is simply this: if everything is infinitely divisible, then the smallest bodies as well as the largest will be composed of an infinite number of parts, and there will be no difference between them.
>
> This has been said to be directed against the Stoics.[20]

There seems to be something disquieting about a set with a proper subset "as big as" itself. It seems absurd that the very largest of things is the same in number of parts as the very smallest of things. Lucretius rejects this absurdity in favor of atomism.

Arnauld (1612-94) and Infinite Divisibility

By the time Arnauld published The Port-Royal Logic, many arguments were taken for granted to "prove" the infinite divisibility of extension. Arnauld cites three from geometry himself.

> 1. Geometry demonstrates that there are certain pairs of lines which do not have a common measure and are called for that reason "incommensurables". An example is the diagonal and the side of a square. If the side of a square and the square's diagonal were each composed of a certain number of indivisible parts, one of these parts would be the common measure of the two lines. Since there is no common measure, it is impossible that the two lines be composed of any number of indivisible parts.
>
> 2. Geometry also demonstrates that although there is no square of a number which is twice the square of another number, still it is quite possible that the area of one square be twice the area of another square. If these two squares were composed of a certain number of ultimate parts, then the larger square would contain twice as many parts as the smaller one; and since both figures are squares, there would exist a square number double another square number -- an impossibility.
>
> 3. Finally, nothing is clearer than this principle: Two entities of zero extension taken together still do not have any extension; that is to say, an extended whole has parts. Take any two of these parts which we assume to be indivisible. I ask whether the parts have extension. If they do not have extension, they have zero extension and the two taken together cannot have extension; if the indivisible parts have extension, they have parts and are hence divisible.[21]

The first argument depends upon the Pythagorean theorem ($A^2 + B^2 = C^2$). The length of the diagonal is computed as the positive square root of the sum of the squares of the other two sides ($C=\sqrt{A^2+B^2}$). Suppose the sides of the square are each of length 1. Then the length of the hypotenuse is the square root of 1 squared plus 1 squared, or the square root of 2. It is a simple reductio proof to show that the square root of two is not a rational number.[22] The first argument is valid, but the premisses are not all true. The argument assumes that the Pythagorean theorem applies in the case of discrete metrics, a premiss that is not true. The Pythagorean theorem formula is derived using the premiss that extension is infinitely divisible. (See chapter VII for a detailed demonstration of the dependence.) Consequently, using the Pythagorean theorem in this context begs the question by presuming what the argument purports to prove.

Moreover, a geometric device for dividing a line of unknown length into a fixed number of equal parts using parallel lines and line segments of known fixed length shows commensurability and is illustrated in figure 3. The ratio of the respective segments is not 1:1. But the perspective ratio[23] for different directions in a discrete metric space is not 1:1 either.

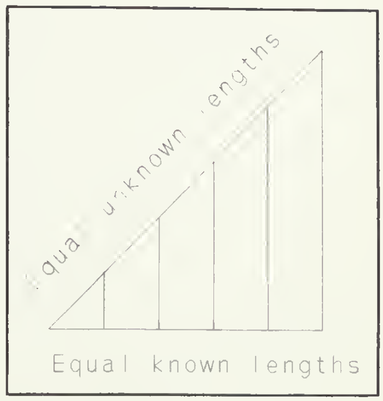

Figure 3. Dividing a line of unknown length into equal parts.

The second argument suffers from a similar fate. It assumes that every line has a midpoint. Unfortunately, in discrete metric spaces, not every line has a midpoint. A line with an even number of points does not have a midpoint; only a line with an odd number of points has a midpoint. Moreover, the perspective ratio varies as lines are rotated in the plane; lines at "45°" have a 1.414:1 perspective ratio and may have fewer points than a line of the same (continuous) length. To illustrate the difficulty consider a square inscribed inside another square as shown in figure 4.

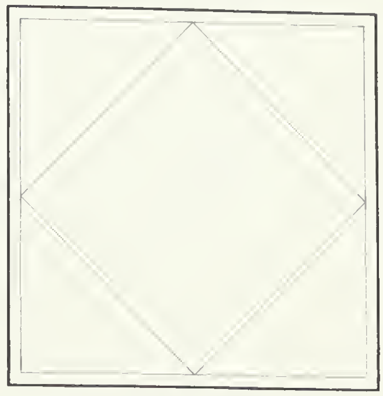

Figure 4. Continuous Metric inscribed squares.

Consider the same diagram using a discrete metric as illustrated in figure 5. I shall select a size which is odd and has many points. The outer square is 7 points long and has an area of 49 points; the inner, rotated, square has an area of 25 points, although each side has only 4 points.

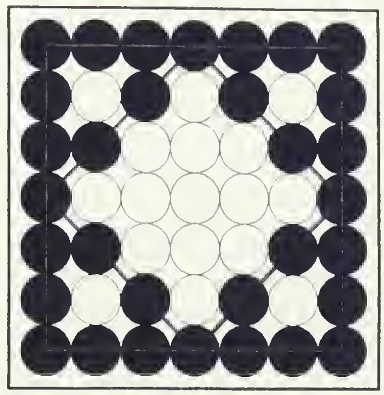

Figure 5. Atomic metric inscribed squares.

One may conjecture that as the size of the outer square gets large relative to the size of a point, the ratio of the size of the outer square to the inner square approaches 2:1. A square with a side 3 points long and an area of 9 points has an inscribed square with an area of 5 and sides of length 2 points, but a square with only two points on a side has no inscribed square at all.

The third argument, if not outright self-contradictory, merely asserts that to have extension is to be divisible: "if . . . indivisible parts have extension, . . . they . . . are . . . divisible".[24]

Arnauld goes on to add another alleged proof. His demonstration imagines a flat (Euclidean) sea with a ship that is receding in the distance. He constructs a similar triangle argument using the eye of the observer, the light ray coming from the waterline of the ship, one coming from the horizon, and an interceded parallel transparent glass. Figure 6 shows the geometry involved. One is supposed to be convinced by this argument that there is a

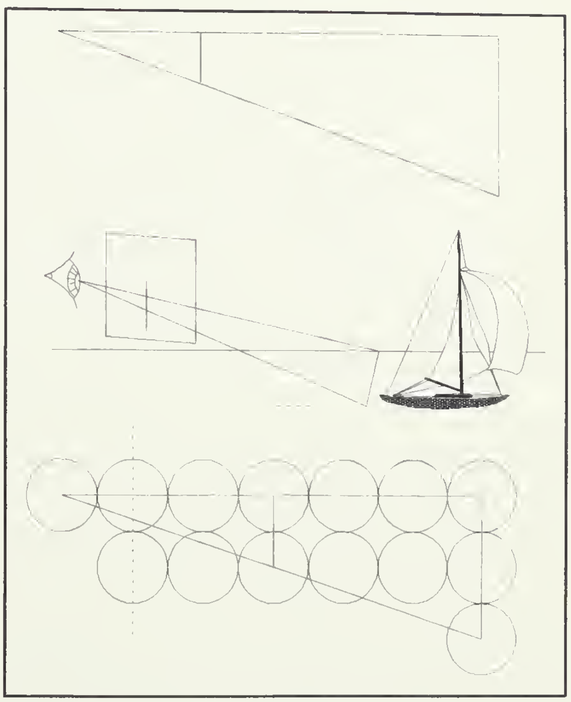

Figure 6. Similar triangle perceptual argument.

point in the plane of the glass where the light ray coming from the waterline of the ship passes. The illustration shows that the passage of the "continuous" line goes through actual points in the interceded parallel plane only when the distances are fortuitously correct. Rays that do not intersect points can be called "virtual" lines. Every line requires at least two points, but the remainder of any line may be virtual rather than actual. The ray that intersects the transparent glass does so at a virtual point rather than at an actual one. When extension is measured with a discrete metric, the experiment proposed by Arnauld results in a hypotenuse which gets "thinner" and "thinner". It does so by passing through only as many points as are between the water line and the horizon. The line is so nearly parallel to the horizontal baseline that it extends for long distances without passing through actual points. The perspective ratio is nearly infinity. One cannot "pick out" two points in the transparent glass. Once one line intersects at a point, the other is "too close" to hit an adjacent point and is only a virtual intersection. The computer implementation shows only a single row of dots close to the pane, as the triangle in figure 7 shows.[25,26]

Figure 7. A long thin triangle only two atoms high on one end.

Hume and Infinite Divisibility

Hume argues against infinite divisibility; he interprets space or extension in terms of the objects which might occupy it. A consequence of the Humean view would be an empirical answer to the question whether extension is infinitely divisible or not.

Hume's argument involves treating space or extension as not distinct from matter. According to Baxter, Hume's view subsumes matter and extension into one structure.

> Hume believes that our idea of a region of space is an abstract idea of the following sort: We think of a region by thinking indifferently of various objects that could occupy that region . . . . [The] upshot of this theory is that regions of space have the structure of extended objects.[27]

Under this view, space or extension would have the same structure as matter. Consequently, the findings of modern physics would be doubly relevant. What we learn about matter is immediately generalizable to extension (space). Baxter essentially showed that Hume believes that the smallest things have no extension and that extended things are created by combining these unextended smallest things.[28]

Modern physical theory corroborates this Humean view. According to current theory, all matter is mostly empty space; extended objects are created by combining objects which have (nearly) no extension of their own. The smallest particles are called 'quarks' and are described by physicists as "point-like" entities.

> The first evidence for the existence of quarks came about 15 years ago [1969] in experiments that probed nuclei with energetic electrons. They revealed point-like objects (then called partons or quark-partons) inside the neutrons and protons of the nuclei.[29]

The influence of these point-like particles "extends" space and matter and produces extended particles (including protons and neutrons). However, whether matter is infinitely divisible or not is still not settled.

> Underneath the standard [physical] model [of matter] is the realm of "compositeness". The standard model holds that everything is built out of six kinds of quarks and six kinds of leptons, and that these quarks and leptons are the most elementary forms of matter. Up to now, whenever physicists have thought they had reached the most elementary constituents of matter, they have been proven wrong. There is a fraction of theorists who think they are still wrong. Believers in compositeness say the quarks and leptons are themselves composite, made of more elementary objects, which may be called preons or technicolor quarks or something else.[30]

Hume's approach would allow the question of infinite divisibility to be settled by empirical physics. But Hume's tack represents a significant departure from Aristotle's thinking. Even if modern physics did provide a definitive answer, Aristotle's difficulties would not be disposed of. Aristotle clearly distinguishes between place and matter in The Physics: "Place is no part of the thing." (211a1). As a result, Hume's approach could be conceived of as predating Aristotle.

Notes and References

1. David J. Furley, Two Studies in the Greek Atomists, (Princeton: Princeton University Press, 1967), p. 7.

2. Furley, p. 7.

3. Furley, p. 8.

4. After one bisection the parts are half the size of the original and are both extended. If a part after N bisections is extended, then half that is still extended. The parts are extended after one bisection, and if the parts are extended after N bisections, then they are also extended after N+1 bisections. Mathematical induction concludes that the parts are extended after K bisections for all K.

5. Furley, p. 10.

6. Furley, p. 13.

7. Furley, p. 13.

8. Furley, p. 13.

9. Furley, p. 14.

10. Furley, p. 16.

11. Furley, p. 17.

12. Furley, p. 17.

13. Furley, p. 19.

14. Furley, pp. 17-18.

15. Furley, p. 18.

16. Furley, p. 18.

17. Furley, p. 22.

18. Furley, p. 25.

19. Furley, pp. 36-37.

20. Furley, p. 36.

21. Arnauld, Antonie, The Art of Thinking: Port-Royal Logic, trans. James Dickoff and Patricia James, (New York: Bobbs-Merrill, 1964), p. 299.

22. Such a proof might go as follows. Suppose $\sqrt{2}$ is rational. Then there exists numbers P and Q such that $\sqrt{2} = P/Q$ (1). We may suppose that the fraction P/Q is expressed in lowest terms. This would require that P and Q are relatively prime, that is, that they have no greatest common divisor (GCD) (2). Now then, squaring both sides gives us the equation $2 = P^2/Q^2$ or that $Q^2 = 2 \cdot P^2$ (3). Since the right hand side of the equation is divisible by 2, so must the left hand side be. But for that to be possible, Q must itself be divisible by 2, and Q must be of the form $2 \cdot R$ (4). Consequently Q^2 $(= 2 \cdot P^2)$ must equal $(2 \cdot R)^2$ (5). This allows us to conclude that P must also be divisible by 2 (8,9) contradicting the assumption that $\sqrt{2}$ could be expressed as a rational number in lowest terms.

(1)	$\sqrt{2} = P/Q$.	Assume $\sqrt{2}$ is rational.
(2)	GCD(P,Q)=1	P/Q is expressed in lowest terms
(3)	$Q^2 = 2 \cdot P^2$	Square both sides & multiply by Q^2
(4)	$Q = 2 \cdot R$	Both sides must be divisible by 2
(5)	$(2 \cdot R)^2 = 2 \cdot P^2$	Substituting
(6)	$4 \cdot R^2 = 2 \cdot P^2$	Expanding
(7)	$2 \cdot R^2 = P^2$	Divide both sides by 2.
(8)	$P = 2 \cdot S$	Both sides must be divisible by 2.
(9)	GCD(P,Q)=2	Both P and Q divisible by 2.
(10)	$\sqrt{2} \neq P/Q$	By reductio $\sqrt{2}$ cannot be rational.

23. See note 8 of chapter VI and page 174.

24. Arnauld, 1964. p. 299.

25. N. Kretzmann, ed., Infinity and Continuity in Ancient and Medieval Thought, (Ithica, NY: Cornell University Press, 1982).

26. J. D. North, "Finite and otherwise: Aristotle And Some Seventeenth Century Views", in Nature Mathematized vol. 1., University of Western Ontario Series in Philosophy of Science, no. 20. William R. Shea, ed., (Dordrecht, Holland and Boston, U. S. A.: D. Reidel, 1983), 113-148.

27. Donald L. M. Baxter, "Hume on Infinite Divisibility", History of Philosophical Quarterly 5 (April 1988): 133.

28. Baxter, pp. 134-5.

29. D. E. Thomsen, "Atomic nuclei: Quarks in leaky bags", Science News Magazine Vol. 125, No. 18, May 5, 1984, p. 297.

30. Dietrick E. Thomsen, "Experimenting With 40 Trillion Electron-Volts", Science News Magazine Vol. 132, No. 20, November 14, 1987, p. 315.

CHAPTER VI

THE VALIDITY OF BOTH VIEWS

In this chapter I present a consistent model for each of the two positions. The traditional number line is developed and examined for the purpose of contrast with the less familiar view -- atomism.

The Validity of Divisionism

Flour is finely grained stuff which has been being divided for millennia. Have you ever tried to measure a cup of flour? One cup, more or less, is likely to differ from another by minute amounts. Water is another divisible quantity. While the grains of flour can be made visible by a sufficiently strong magnifying glass, the "grains" of water cannot. True, we have obtained electron micrographs which seem to show the individual atoms of some of the heavier metals. But the same still cannot be said of water.

We are accustomed to dividing "stuff". To assist in dividing stuff, we have devised units of measure. We have invented arbitrary units of measure and conversions between them. For example, to convert from liters to gallons we multiply the number of liters by 0.2641720524.[1] The process of dividing materials, in conjunction with measuring how much (quantity), has developed into the use of numbers for measuring and dividing.

Developing Numbers

The fifth or sixth century equivalent of a Certified Public Accountant kept track of inventories. One may reasonably presume that counting filled containers along with adding tallies was sufficient for inventory purposes. But, when the auditors came along, subtraction was necessary to determine how much there "ought to have been". Tallying the last inventory, adding the records of the amounts received, and subtracting the records of the amounts drawn from stores would yield how much a current inventory should find.

Negative Numbers

With the advent of subtraction it is only a matter of time before negative numbers are needed. When the plans for issues from stock, say in planning for a battle or a trip, are compared with the inventory and found wanting, this wanting can be quantified by negative numbers. "We need 100 sacks of flour more", the planner might say. The answer to "How many will we have after the trip?" becomes "minus 100".

Inventing multiplication merely needed an extremely rich person or a ruler with too many inventories to tally directly. Adding the same size tally many "times" yields how many "times" we have that tally. It isn't easy to tell the ruler with 45 granaries and a tally of 150 sacks of flour in each granary how many sacks of flour he has. We soon learned how to multiply 150 "times" 45.

Fractions

Division follows quickly with the need to allocate stocks evenly to a number of storage locations, or to the legions of soldiers. How much do we have? How many battalions of legionnaires must we divide this among? Finding out how many supplies to take away and give to each battalion is a long and cumbersome process when trial and error subtraction is used. The bigger the bureaucracy, the more the need there is for division as a tool. With a notation for numbers in place, it's just a matter of time before someone devises a better way; division is that way. But when the answer doesn't come out even, the result is a fraction of a whole. Since simple fractions (dividing in two, etc.) were not unfamiliar, that division sometimes yields them makes it less strange.[2]

Pure Numbers

It falls to the philosophers to analyze these relations and develop a theory of pure numbers -- numbers which are viewed apart from that which they might be applied to in measuring or counting. The '3' in '3 bags of flour' is seen as something existing apart from the bags of flour.

There is a natural progression in this development. First there are tally strokes. Tally strokes are measured by numbers which count the tally strokes. Addition of numbers becomes a shorthand for tallying a lot of tallies. This process generates the natural numbers. Whenever any two natural numbers are added, the result is a natural number. The natural numbers are said to be *closed* under addition. If something is added to something else and a result is obtained, then one ought to be able to subtract something from the result and obtain something else. This process works well enough for some pairs of natural numbers, but not for other pairs. Three less one gives two, but one less three does not give any known natural number. It gives the number which, when added to two, gives zero. We extend the natural numbers to include this zero as well as the other strange numbers, which we call the negative integers.

When we include these negative integers, zero, and the positive integers and thereby obtain simply the integers, we find that the integers are closed under both addition and subtraction.

Multiplication

We found that adding a natural number to itself many times is tedious and invented or discovered a shorthand -- multiplying that natural number by the number of times we took it as an addend. We found that the natural numbers are closed under multiplication. We also found that the integers are closed under multiplication. One strange effect was noted -- the product of two negative integers is a positive integer.

Division

Naturally, if one thing is multiplied by something else and a product is obtained, one might well ask how the one thing might be obtained from the product and something else. Since a product is obtained by adding one thing many times, the one thing could likewise be subtracted many times -- distributed among the many somethings or divided among them. In such a case the one thing is called the *divisor* and the other factor, which when multiplied by the divisor gives the product, is called the *quotient*.

A similar problem to the one which arose in subtraction arises. For some pairs of numbers, taking one as a product and the other as a divisor, dividing the divisor into the product yields a known integer. For example, six divided among (by) two gives three. But for some of these pairs of numbers, taking the first as a divisor and the other as a product, dividing the divisor into the product does not yield a known integer. For example, two divided among six does not yield any known integer, positive or negative. Each of the six only gets a "fraction" of a whole. Initially, fractions, as such new numbers are called, are added to our growing list of types of numbers. Combining fractions with the integers gives what we call rational numbers. We notice one more anomaly. Zero cannot be divided into other numbers. It seems an ad-hoc solution, but we just forbid division by zero. It doesn't work.

I've given a hypothetical account of how numbers might logically have been developed. For the purpose of this work, rational numbers are all we need. Let's look at the properties we pretty much take for granted of the rational numbers. Whenever any two rational numbers are added, multiplied, subtracted, or divided (excluding the forbidden division by zero), the result is another rational number. The rational numbers are closed under the four arithmetic operations -- addition, subtraction, multiplication, and division.

Order

The relation "less than" (or "greater than") has been taken largely for granted. Higher counts are greater than lower counts. Using this relation, it is found that any two number must satisfy one of three relations (the trichotomy).

1. The first is less than the second.
2. The first is equal to the second.
3. The second is less than the first (the first is greater than the second).

When a number is greater than a first number and less than a third number it is said to be "between" the other two. It is easy to show that given any two distinct rational numbers, it is possible to find another between the other two. For example, consider the rational numbers 1/3 and 1/2. 1/3 is the same as 4/12; 1/2 is the same as 6/12. Clearly 5/12 is between 4/12 and 6/12. Since 4/12 is 1/3 and 6/12 is 1/2, 5/12 is between 1/3 and 1/2. As a general procedure, one may add 1/2 the difference between the numbers to the smaller number. This will always yield a number less than the higher number and larger than the lower number, or <u>between</u> the two given numbers.

The property of having another element between any two given members is called *denseness*. A set of numbers is *dense* if and only if between any two numbers in the set there is another number in the set.

The natural development of numbers for counting and measuring originally had the numbers intimately associated with the stuff being counted or measured. At these practical levels numbers were never separated from the things they were a measure of. It was only with the invention of "pure" numbers, numbers taken apart from the things they were traditionally used to count or measure, that the properties of numbers could be separated from the properties of the things they were used to measure. And the properties of numbers drives the questions about the properties of the stuff they are used to measure.

Extension Without Measure

When rational numbers are used in measuring the quantity of stuff, it is presumed that the stuff is as divisible as are the numbers. More in question is the so-called "extension" of stuff rather than its matter. The rational number system we use to measure "how much" (stuff) with has the property of denseness. We can continue dividing between numbers as long as we like. The question that arises naturally is "can the stuff we associate the numbers intimately with be similarly divided?" Even talk of (pure) extension dissociates the "distance across" some stuff from its matter. When we remove the matter from consideration we talk of pure "extension".

We are accustomed to measuring extension relative to directions. Two (non-parallel) directions are required to measure area; three are required for volume. We can certainly conceptualize the (empty) space as something apart from the system of numbers we use to measure it. But when we ask "how much" in regard to such extension, we talk about its "measure" -- the numbers we use to describe how much. Talk of divisibility also asks after the "stuff", including the empty space, we use the numbers to measure.

We are posed with a complex question. We have a consistent model for measuring divisibility, one that exhibits denseness, and can therefore support infinite divisibility. We also have only conceptualization as a way of holding onto the concept of the extension of empty space (or of matter). But our conceptualization is amenable to using the model provided by numbers. We conceptualize the difference between two distinct points in a visualized blow-up where we can picture ourselves walking (part way) from one point to the other. So we naturally argue, by analogy, that "pure extension" is similarly divisible. One point of the present work is that, sufficiently informed, the analogy is not so obvious.

In the following section I present a much less familiar model, one which presumes that the model provided by the rational numbers does not apply, Once that model has been presented it will no longer be such an "intuitively obvious" conclusion that the "stuff" of empty space is best modeled by the rational (or real) number system. The fifth postulate of Euclidean geometry was once thought so intuitively obvious it was taken to be a "self evident" truth, yet we now know of self-consistent geometries based on "less intuitive" statements of the postulate.

The Validity of Atomism

In this section I build and examine a consistent model which is based upon the premiss of an indivisible minimum extension. Computer graphics screen displays implement this model.

> But although most geometricians from the time of Euclid have in fact worked with the principle of infinite divisibility, mathematicians do not refuse to consider the possibility of a geometry of finite divisibility.[3]

A model of extension using atomic magnitudes, while not consistent with infinite divisibility, is not inconsistent by itself. An analogy with geometry will serve to illuminate my view. For millennia people tried to prove the insight that parallel lines never meet -- the so-called Euclidean or fifth postulate. More recently the fifth postulate was shown to be independent of the others. This independence allows constructing a variety of non-Euclidean geometries. They are each internally consistent but generally not compatible with each other. Each geometry depends upon the form chosen for the fifth postulate. In a similar manner people have argued for the intuition that extension is infinitely divisible. As in the history of geometry, in which various arguments were thought to prove the Euclidean form of the fifth postulate, various arguments have been advanced as refuting atomism. Two modern views belie these historical alleged refutations of atomism.

One view is that provided by the invention of the microscope. Microscopically granular substances appear continuous at macroscopic levels. Modern particle physics has found a hierarchy of successively smaller particles cumulating in quarks and leptons which are indivisible -- so far. Matter, strictly speaking, is not extension, but the extension of matter appears to be quantized (atomic), although at smaller levels than the namesake. Extension per se can be quantized with a discrete metric.

The other view derives from the advent of computer graphics. Computer display screens exhibit de facto atomic extension. The smallest portion of a display is called a pixel (picture element). IBM PC Monochrome Graphics Adapter (MGA) and Color Graphics Adapter (CGA) displays have 640 horizontal by 200 vertical pixels. Enhanced Graphics Adapter (EGA) displays have 640 by 350 pixels. Vector Graphics Adapters (VGA) displays have 640 by 480 pixels. Drawing a line on one of these graphics displays requires turning on successive or contiguous pixels. A minimum length line consists of two adjacent pixels (points). Except on very high resolution displays, lines not aligned with the axes appear as small step functions. Even on very high resolution displays, lines appear as step functions when viewed through a magnifier.

Now that computer displays have become commonplace, they may be used as an example for illustrating discrete metrics. By their very nature they implement discrete metrics, and they serve as a good example to illuminate a certain "weirdness" inherent in discrete metrics.

Let a display be specified as composed of an NxM array of pixels. Drawing a square on such a field requires using the same number of pixels across as up and down. Drawing the diagonal, however, uses only one pixel for each of the vertical and horizontal positions. There are exactly the same number of pixels in the diagonal as there are in both the horizontal and the vertical sides. Figure 8 is an example of a 7x7 square with one diagonal drawn on such an array.

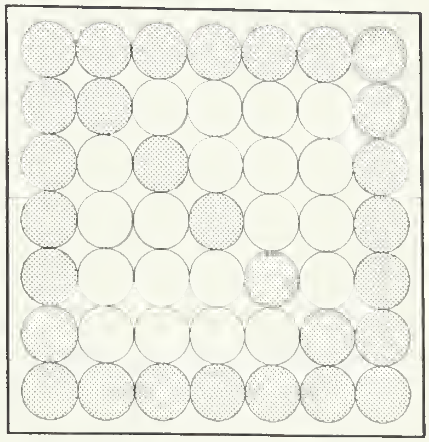

Figure 8. A 7x7 square with one diagonal drawn on an atomic array.

The usual metric with the Pythagorean criteria of preserving distance with any rotation cannot be presumed to hold. The distance between adjacent pixels is 1 minimum unit, no matter what the direction. The shape of the pixel affects the "size" of the distance according to some <u>external</u> criterion. Within the system, there is no way to discern that the diagonal distance differs from horizontal or vertical distances. I shall discuss these differing "sizes", but I shall have to deal with a few preliminaries first.

In addition to squares, triangles and hexagons also form plane-filling patterns using pixels which exhibit some kind of radial symmetry. Hexagons seem likely choices as they approximate circles or "dots". There are "more" directions which have the same "distance" (0°, 60°, 120°, etc.). Hexagons allow 6 directions of symmetry, but unfortunately, hexagons cannot "slide" past one another, while triangles and squares can. Also, hexagons are made up of smaller triangles. Triangles allow three directions of symmetry, while squares allow two.

When it comes to three-space, cubes form space-filling solids. Equilateral pyramids do also. Cubes allow three directions of "sliding", while pyramids allow four.

But specifying a position in two-space requires only two coordinates; similarly, specifying a position in three-space requires only three coordinates. Consequently, the simplex pattern (triangle, pyramid) has a redundancy in its directions of movement resulting in a loss of 1 degree of freedom. Parsimony is achieved by requiring orthogonality in each additional dimension. Squares and cubes are therefore the logical choice for conceiving atomic pixels. However, using squares for illustrations prevents distinguishing individual pixels. To allow distinguishing individual pixels, I shall use circles for illustrations.

We are not accustomed to thinking of extension in terms of atomic distances. In fact, we are so accustomed to thinking of extension as being continuous that we have difficulty even conceiving of it as being atomic. One branch of mathematics which covers theories of distance is metric space theory. A theory of distance which has integral units of distance is called a discrete metric. The distance from a point to itself is always zero in a metric space, discrete metrics included. Under an atomic theory of extension, there is no zero unit of extension, although there is a zero unit of distance. The measure of distance between two points is different from the extension or length of the line that includes those two points. The distance from a point to itself is always zero, even though the extension of the point itself is not. The distance from a point to its nearest neighbor is one unit of extension, but the length of the line including the two points is <u>two</u> units of extension.

The length of a line composed of only two points is actually a *minimum* of two units of extension. The continuous length of such a line can be larger than two units of extension. The ratio of the length of such a line to one with the minimum extension may not be 1:1. One can conceptualize these differences in "distance" using the concept "aspect ratio".

Aspect Ratio

The ratio of the horizontal and vertical distances is usually not one to one on computer displays. For example, my EGA display on a 13" monitor has 640 pixels wide by 350 pixels high displayed in an area which is 10 inches wide by 7.5 inches high. The ratio of the horizontal height to the vertical width of a picture or screen is called the 'aspect ratio'.[4] On my display, which is 10 inches wide by 7.5 inches high, the aspect ratio is 10/7.5, which is 1.33 or 4:3.

Drawing lines on such a display is also affected by the shape of pixels themselves. In the case of my EGA display, there are 640 pixels in 10 inches horizontally and 350 pixels in 7.5 inches vertically. The horizontal size of a pixel is 10"/640. The vertical size of a pixel is 7.5"/350. The ratio of these is 0.729 -- 0.729 = (10/640)/(7.5/350) or 35:48. The screen itself is wider than it is tall, but each pixel is taller than it is wide. These facts must be taken into consideration when drawing pictures on such displays. If one presumes that the aspect ratio of a pixel is 1:1

when one draws pictures on such a display, the resulting pictures will appear distorted.

When a picture is stored or recorded using one aspect ratio and reproduced using another aspect ratio the resulting view will also appear distorted. This effect can be illustrated by a familiar technology in film. Wide screen, or cinemascope[5], motion pictures do not use wider films to store and project the wider picture.[6] Cinemascope pictures use standard 35 millimeter films. How can the wider picture be stored on the film? A special lens, called an anamorphic lens, is used which distorts the image on the film by shrinking it in the horizontal direction.

> The aspect ratio of the cinemascope frame is 23.8:18.67 = 1.275 instead of the normal 1.38. Since the anamorphic system operates with a ratio of 1:2 the effective screen aspect ratio will be $(23.8/18.67) \cdot 2 = 2.55$.[7]

The image of a square will appear on the film as a rectangle which is narrower by half than it is tall, and the image of certain double wide rectangles will appear on the film as squares. The image is stored on the film with a "perspective ratio"[8] which is not 1:1. Because the horizontal compression is twice the vertical compression, the perspective ratio on film is 1:2. When a cinemascope motion picture is projected a special anamorphic projection lens is used to widen the image back to its original proportions.[9]

A picture recorded using a cinemascope lens will be stored with a perspective ratio which is 1:2. If that picture is projected using a standard lens it will be projected presuming a perspective ratio of 1:1. A cinemascope picture projected using a standard lens will appear squeezed together and too tall. I have seen Cinemascope pictures appear this way on television while the credits are running. Conversely, a standard picture projected with a cinemascope lens will appear stretched out and too short. A standard film is stored with and projected with a 1:1 perspective ratio. A cinemascope film is stored with and projected with a perspective ratio which is 1:2. Each picture will appear normal when it is projected using the same perspective ratio with which it was stored. But when either picture is projected using a different perspective ratio from that with which it was stored, it will appear distorted.

Anyone who has tried to draw a low resolution picture on a computer screen using a character, say the asterisk, can see the effect immediately. A square number of asterisks, say 6x6, hardly looks like a square. And when one finally gets something that looks reasonable on the display screen, it looks different when it is printed.

The aspect ratio of most display screens for characters is about .42.[10] To make a square using six lines of text one would need 6 divided by .42 or 14.4 characters on each line. But what looks like a square on the screen prints out as too wide. The aspect ratio for printed text in a 10 characters per inch by 6 lines per inch mode is .6.[11] The printed square using six lines of text would require a width of only 10 characters. In figure 9, the left array is numerically square. On the screen the middle one looks square, and on the printed page the right one looks square.

******	**************	**********
******	**************	**********
******	**************	**********
******	**************	**********
******	**************	**********
******	**************	**********
6 x 6	14 x 6	10 x 6
1/1=1	1.8/.75 = 2.4	1/.6 = 1.67
6x1=6	6x2.4 = 14.4	6x1.67 = 10

Figure 9. Square, Display, and Print aspect ratios.

Measuring distances using a metric based on the atomic premiss requires that any "distance" be in terms of multiples of the minimum distance, the distance between adjacent points. The concept of perspective ratio can be adapted to give us a quantitative measure of the difference, in <u>non-a-</u>

tomic terms, between the scales used for different directions. Pixel aspect ratios show that one perspective can illuminate (or obfuscate) the other, as when the square on the display screen doesn't print square. But one perspective (atomic or continuous) cannot be used to evaluate the other lest a contradiction be introduced in the overall system. Like the wave particle duality of matter, transformation equations must be rigorously (and religiously) used when switching perspectives.

Under the atomic presumption there are adjacent points, and the minimum length of a line segment consisting of two adjacent points is two minimum units. We can only visualize these points as "dots" of a fixed size and indeterminate shape. We would like to presume that these dots can be thought of as small disks and will do so for illustrative purposes, but in consideration of the foregoing discussion of the aspect ratio of pixels, we must be ready to cast this assumption aside. Some argue that this assumption might entail a contrary presumption -- that angle is infinitely divisible, a question dealt with elsewhere.

Drawing lines in discrete metric spaces

A line between two points may have two sides. Figure 10 is an illustration on a 7x9 array of pixels. The line from point A to point B passes directly through point O but passes immediately to the right of points represented by a left semicircle and to the left of points represented by a right semicircle.

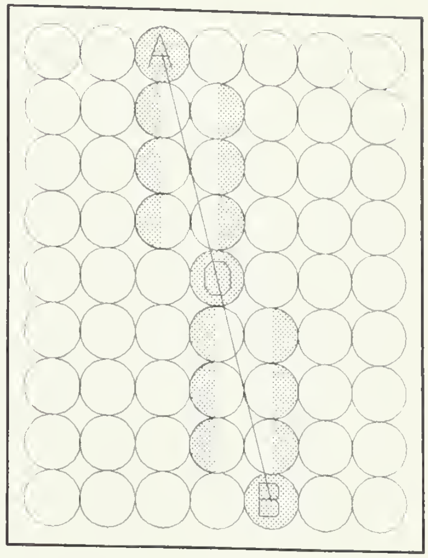

Figure 10. The two sides of a line.

A closed plane figure would have to include either the left semicircle points or the right semicircle points depending upon on which side of the line the figure was. Both figures would include points A, O, and B. Obviously, the degree of overlap depends upon the orientation of the line as well as its length. As a practical matter, computer implementations of line drawing functions presume a "preferred direction". Turning on the rightmost one of a horizontal pair of pixels and the lower one of a vertical pair is only one of 4 possible implementation strategies and is shown in figure 11.

Before we can intelligently discuss overlapping plane figures, we must examine intersecting line segments. Two lines intersect at a point, and in atomic or discrete metrics a point has a finite size or a minimum length. For the following discussion, lengths (and areas) are given in terms of the minimum unit of size. If the length of two intersecting line segments are A and B, then the length of the combined segment is A + B -1. One point is shared by both line segments, and its size must be subtracted. If A and B were merely added, as we are accustomed to doing with continuous distances, the overlapping point would be counted twice; its length must be subtracted. Figure 12 is an illustration.

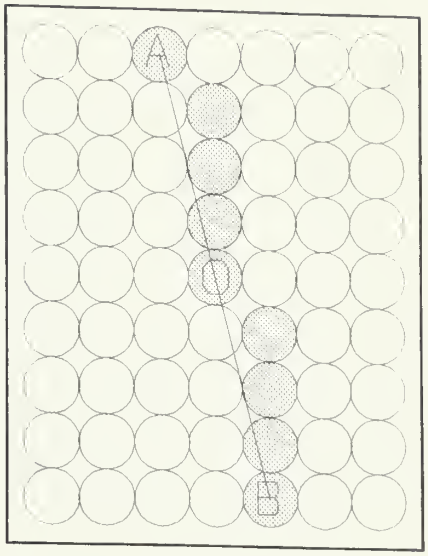

Figure 11. Using the rightmost and lower points to fill out a line.

Segment	A	B		length	2	*	*	
Segment		B	C	length	2		*	*
Segment	A		C	length	3	*	*	*

Figure 12. Adding lengths of line segments.

Of course, there are also line segments which cross each other but which do not share an actual intersection point. Figure 13 shows such a case. Line segment AB has a slope of -2; segment CD has a slope of 1. The positioning of these lines is such that there is no point that both line segments pass through. If the only points the line actually passed through were selected in an implementation, segment AB would appear as a sequence of dots.

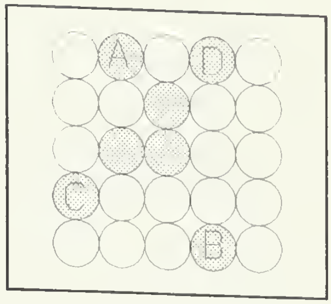

Figure 13. Line segments which cross but do not share a point of intersection.

To "thicken" the line and make it more visible, the points immediately to the left of or to the right of the line must be selected. Figure 14 identifies the points which would be selected were the line thickened "to the left". In both figure 13 and 14, line segment AB "crosses" line segment CD but does not pass through a point on segment CD. In such cases the extension of the two line segments taken together does add up to the sum of the individual extensions. The geometric interpretation of this is satisfied for finite geometries. In figure 15, the line is thickened to the right. This option implements the strategy mentioned illustrated in figure 11 above. In this case one of the points used to thicken line segment AB is one of the points on line

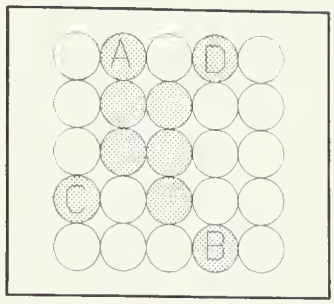

Figure 14. Points selected for thickening a line to the left.

segment CD. There is a shared point of intersection in this case, but it is not located where it would be located were the lines continuous. In such cases the extension of the two line segments adds up to the sum of the individual extensions *less the extension of a single point*.

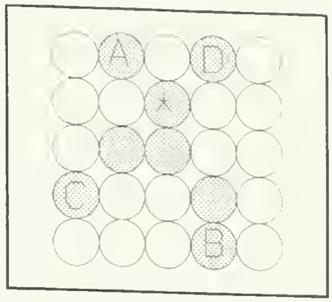

Figure 15. Points selected for thickening a line to the right.

When it comes to drawing plane figures, we must rely on our experience with continuous metrics, but we must be prepared for "weird" (unintuitive) differences.

Constructing triangles in the atomic plane

Let us examine some "minimum sized" triangles with atomic magnitudes. Clearly the smallest has two sides each of length 2 (points). Since the two side lines intersect at a point, one point is shared by both lines. Rotation aside, there are only two ways to draw such a figure. Figure 16 is a degenerate triangle -- a line segment consisting of two intersecting collinear line segments. Figure 17, on the other hand, is a recognizable triangle. Notice that the diagonal has a length of 2 points, hence is

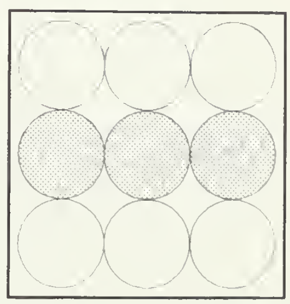

Figure 16. A degenerate atomic triangle.

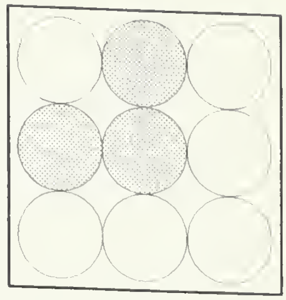

Figure 17. The minimum sized atomic triangle.

the same "size" as the other two sides. However, the perspective ratio between the diagonal and the horizontal (or vertical) is not 1:1. In fact, it is 1.414:1. Perspective ratio relates ratios of atomic distances (number of pixels) to continuous (infinitely divisible) distances along different directions or dimensions. We must beware that we do not evaluate atomic figures from the perspective of presuming infinite divisibility. To do so would be to beg the question or, worse yet, introduce a contradiction, from which anything follows.

All this seems a little strange when held up against our conventional view, which is based upon continuous lines. However, lest we fall into the same trap the ancients did, it behooves us to develop a little familiarity with the atomic perspective.

Drawing a triangle in a discrete metric space so that the length of its sides in continuous distances is comparable to the number of pixels on the line requires some ingenuity. There is a way which can make maximum use of our familiarity with continuous metrics. Locate the vertex points on the centers of pixels as shown in figure 18. Draw

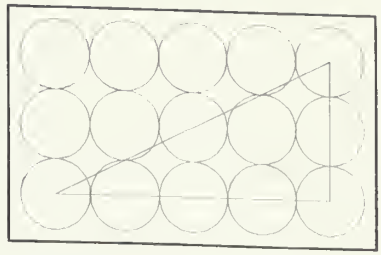

Figure 18. Locating vertex points of a continuous triangle on atomic points.

continuous (divisible) lines connecting the vertexes. Examine the center point of each pixel interior to the continuous triangle or overlapping it. If the center point of the pixel falls on or inside the triangle, then count this pixel as part of the area of the triangle. The illustrations help guide our understanding, but we must develop a formula for the area of a triangle which can be compared to the familiar continuous formula.

Computing Area of an Atomic Triangle

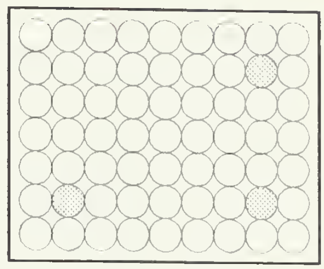

Figure 19. Locating the vertices of a right atomic triangle.

The area of an atomic triangle is not simply $\frac{1}{2}BH$. We can compute the area by devising a mapping from the continuous plane to the atomic plane. Here's how. Draw a right triangle with the vertices centered on atomic points so that the base B and height H cover the requisite number of atomic points as in figure 19. Next fill in the points on the base line and the height line as in figure 20. Then draw the diagonal line connecting the two vertices, and fill in the points which are on or interior to that line as in figure 21.

We can compute accurately the number of points to fill in without actually drawing the line by noting the relation between distance along the base and the proportions of the triangle. We are, in effect, computing the height of each similar triangle which has an integer number of points along the line of the base. In figure 22, the smaller triangle and the larger triangle are similar. This similarity may be expressed in a precise proportion. Side h is to side b as side H is to side B -- h:b::H:B. The corresponding mathematical formula, $h/b=H/B$ allows us to compute side h; $h=(H/B)\cdot b$. Even though the extension of the sides is the number of points and is given by B and H, the length in continuous distances is actually B-1 and H-1. (The starting point is not counted in measuring distances, but must be counted in measuring the atomic extension of the sides.) If we count out to the I^{th} point along the line of the base, we can compute how many points are under the (continuous) diagonal by using

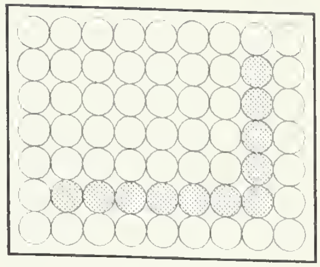

Figure 20. Filing in the base and height of an atomic right triangle.

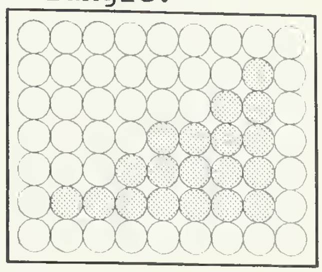

Figure 21. Filing in the interior points of an atomic right triangle.

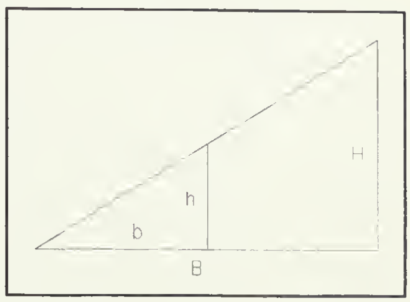

Figure 22. Similar continuous triangles.

the proportion in continuous distances. Because of the relationship between length and extension, the actual length of the base of the smaller continuous triangle is 1 less than the number of points. The length of the base of the triangle with I points is I-1. The length, in continuous distance, of the vertical line under the diagonal is computed using the appropriate proportion of the height. When the appropriate values are inserted the atomic version of the formula becomes $((H-1)/(B-1))\cdot(I-1)$ or $(I-1)(H-1)/B-1)$. If we take the integer part of this we will have the length in atomic distance units of the height of the triangle with I points along the base. But the extension of that line is one point more than its continuous length. The expression for the extension of this line is INT((I-1)(H-1)/(B-1))+1 -- which is just the total number of points in the vertical column of points at the I^{th} point along the base. Adding those extensions for each point along the base gives us the total area of the triangle in atomic points -- the SUM from I = 1 to B of INT((I-1)(H-1)/(B-1)+1).

The area of an atomic triangle with base B and height H is $\sum_{I=1}^{B} \mathrm{INT}((I-1)\cdot((H-1)/(B-1))+1)$; it is not simply $\frac{1}{2}BH$. Figure 23 is a table of the areas of discrete triangles up to 10x10. The values are computed using the above formula. It can be shown that this formula reduces to the familiar

$\frac{1}{2}BH$ as the size of the individual points approaches zero.[12]

Height	Area of atomic triangles								
10	11	16	22	26	31	37	41	46	55
9	10	15	19	25	28	33	37	45	46
8	9	13	17	21	25	29	36	37	41
7	8	12	16	19	22	28	29	33	37
6	7	10	13	16	21	22	25	28	31
5	6	9	11	15	16	19	21	25	26
4	5	7	10	11	13	16	17	19	22
3	4	6	7	9	10	12	13	15	16
2	3	4	5	6	7	8	9	10	11
Base	2	3	4	5	6	7	8	9	10

Figure 23. Table of areas for selected atomic triangles.

I have developed a consistent mathematical formula, but it is not always clear exactly what the drawings look like when compared to corresponding continuous figures. Figure 24 shows various small discrete (right) triangles. Tabulated with each one is its size and the lengths of the opposite and adjacent sides (A and B).

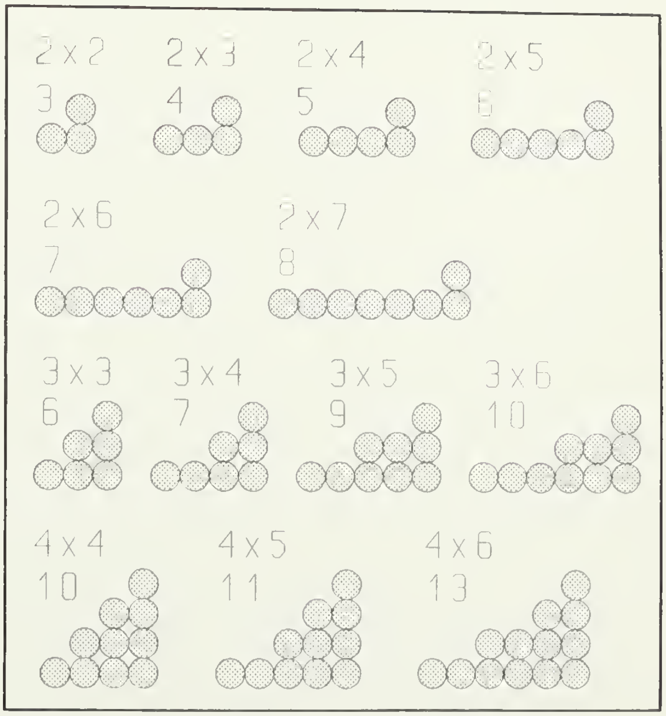

Figure 24. Various small atomic triangles.

Overlapping Atomic Figures

Armed with some familiarity with intersecting (and non-intersecting) lines and simple triangles, we are now in a position to examine overlapping plane figures. Consider the square with a diagonal in Figure 25.

Drawing a diagonal across a square usually divides the square into two triangles. It is no different in atomic metrics. But, because there is a minimum extension in atomic metrics, the line that forms the diagonal is itself extended.

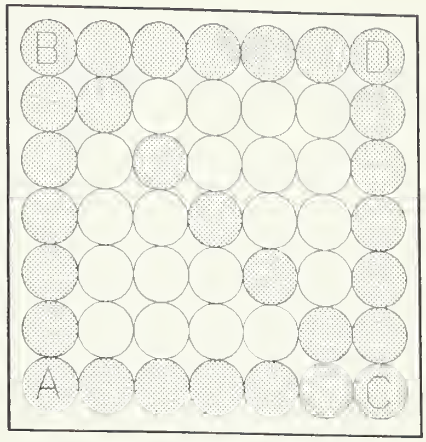

Figure 25. A square with one diagonal.

This line is a part of both triangles; its extension is therefore included in the extension of each of the two triangles. Consequently, the area of the two triangles, which includes the area of the line twice, is larger than the area of the square. The area of triangle ABC is 28 units of extension. The area of triangle BCD is also 28 units of extension. But the sum of the areas of triangles ABC and BCD -- 28 + 28 = 56 -- is larger than the area of the square -- 7 x 7 = 49. The area of their common line, BC, is 7. The area of square ABDC is the sum of the areas of triangles ABC and BCD less the area of the common line -- 28 + 28 - 7 = 49.

This property must be taken into consideration very carefully whenever traditional geometric demonstrations are attempted. The areas of adjacent figures do not just add up to the area of the figure they comprise. Any such demonstrations must be reexamined in the light of the atomic perspective, if a corresponding atomic demonstration is to be made. The results of such demonstrations are often different from those in continuous geometries. One such demonstration involves the traditional (non-atomic) "proof" of the Pythagorean theorem; this theorem has figured into alleged proofs of infinite divisibility.

Notes and References

1. William H. Beyer, ed., CRC Standard Mathematical Tables, 25th Ed., (Palm Beach, FL: CRC Press, 1973), p. 3.

2. The "hypothesized" development presented above follows the logical development of number systems rather than the historical development. The historical record suggests that integers were followed by fractions, that a symbol for zero wasn't devised until positional notation was invented, and negative numbers first showed up in Chinese matrix algebra in the third century. Dirk J. Struik, A Concise History of Mathematics, (New York: Dover Publications, 1967).

3. David J. Furley, Two Studies in the Greek Atomists, (Princeton: Princeton University Press, 1967), p. 5.

4. Aspect Ratio: ratio of width to height of the picture image projected on the screen or printed on the film, the height being taken as unity. The long-established film aspect ratio, still retained for narrow-gage film, is 4:3 (1.33:1). The Focal Encyclopedia of Film & Television Technique, (London: Focal press, 1969), p. 50.

5. Cinemascope: Trade name of the most widely used method of anamorphic wide-screen presentation; camera lenses producing images on 35mm film with a 2:1 lateral compression are viewed with compensating horizontal expansion on projection. The Focal Encyclopedia of Film & Television Technique, (London: Focal press, 1969), p. 132.

6. To do so would have required theaters to invest in expensive additional projectors, and the expense would have inhibited the spread and use of the technology.

7. Michael Z. Wysotsky, Wide Screen Cinema and Stereophonic Sound, translated by Wing Commander A. E. C. York, (New York: Hastings House, 1971).

8. While there are no "pixels" on recorded film, we can measure an effective pixel aspect ratio by comparing the ratio of the sides of the image of a true square (as measured by continuous metrics) when it is recorded on film. But since 'effective pixel aspect ratio' is a cumbersome phrase, and the concept will need to be used frequently, I shall coin 'perspective ratio' to use in its stead. I shall also extend the concept to cover any ratio involving two different scales of measurement. The need for this exten-

sion will be apparent, and its use will be immediately appropriate, for discussions involving lines in the atomic plane. While an aspect ratio is the ratio of two distances, perspective ratio, as defined here, is the ratio of two scales for measuring distance.

9. These lenses cost only a tiny fraction of what an entire projector would cost, enabling the spread and use of the technology.

10. Aspect ratio for characters is very similar to aspect ratio for pixels. The difference is that each character position is treated as a low resolution pixel. An 80x25 character screen using my 13 inch monitor (10x7.5) gives a horizontal "pixel" size of 10"/80. The vertical "pixel" size is 7.5"/25. The aspect ratio is (10/80)/(7.5/25)= .42 or 5:12.

11. Aspect ratios for printed characters is also very similar to aspect ratio for pixels. Printed page formats can be measured in terms of lines per inch and characters per inch. The width of a character is simply the reciprocal of these parameters. Ten characters per inch gives a horizontal "pixel" size of 1/10 inches. Six lines per inch gives a vertical "pixel" size of 1/6 inches. The aspect ratio is (1/10)/(1/6) = .6 or 3:5.

12. For this demonstration we note that as the size of each point gets smaller the number of points per inch gets larger. We must transform the equation into an expression using inches rather than points. For this purpose we may let the number of points per inch be K. Then B and H will be the sizes in inches of the sides, and K·B and K·H will be the corresponding number of points. As the number of points per inch, K, gets very large, the contribution of one point to the length gets very small, and any error introduced by dropping the "INT" portion of the formula will become negligible. The formula itself,

$$\sum_{I=1}^{B} \mathrm{INT}((I-1)\cdot((H-1)/(B-1))+1)$$

reduces to:

$$((H-1)/(B-1))\cdot\sum_{I=1}^{B}(I-1) + \sum_{I=1}^{B}1.$$

Transforming the formula so that the result includes expressions for length in inches requires substituting the corresponding number of points expressed in inches times the number of points per inch. To obtain an area result which is in terms of square inches, the point area formula must be divided by the number of points in a square inch, namely K^2. The new area formula becomes:

$$[((K \cdot H-1)/(K \cdot B-1)) \cdot \sum_{I=1}^{K \cdot B}(I-1) + \sum_{I=1}^{K \cdot B} 1]/K^2$$

Performing the summation and some algebra yields:

$$[(K \cdot H-1)/(K \cdot B-1)) \cdot K \cdot B \cdot (K \cdot B-1)/2 + K \cdot B]/K^2.$$

$$[(K \cdot H-1)) \cdot K \cdot B/2 + K \cdot B]/K^2.$$

$$(H-1/K)) \cdot B/2 + B/K.$$

Taking the limit of this as K approaches ∞ yields: $\frac{1}{2}BH$.

CHAPTER VII

CONCLUSIONS

Most of this chapter will be devoted to showing why the Pythagorean Theorem cannot be used in the argument against atomism. Without the Pythagorean Theorem many arguments against atomism fail. The less mathematically inclined may wish to skip ahead to the concluding remarks on page 208.

The Atomic Destruction of the Pythagorean Theorem

One alleged proof of infinite divisibility appeals indirectly to the Pythagorean theorem.[1] In this section I examine a classic proof of the Pythagorean Theorem and show that that proof depends upon and presumes that extension is infinitely divisible. I also show that, when atomism is presumed, the corrected proof fails to yield the well known formula; a different formula results. Since the original Pythagorean formula itself depends upon infinite divisibility, using it to "prove" infinite divisibility, in effect, begs the question.

A Classic Proof of the Pythagorean Theorem

One diagram used in demonstrating the proof of the Pythagorean theorem involves 6 overlapping figures in the other dplane.[2] Figure 26 shows a square with sides of length C that is inscribed inside another square. The sides of the triangles so formed are of length A and B respectively. The area of each triangle is AB/2. The length of a side of the outer square is A + B. To prove the Pythagorean theorem, the total area of the outer square is set equal to the area of the 4 triangles plus the area of the inner square. The proof involves simple algebra.

Figure 26. A square with sides of length C inscribed inside another square.

Pythagorean Theorem Proof (with infinite divisibility)[3]

$$\begin{aligned} (A + B)^2 &= 4\cdot(AB/2) + C^2 \\ A^2 + B^2 + 2AB &= 2AB + C^2 \\ A^2 + B^2 &= C^2 \end{aligned}$$

This proof of the pythagorean theorem depends upon a line having no area and a perspective ratio of 1:1. In the proof the area of contiguous triangles and a square is added and set equal to the area of the overall figure. In order for the area of contiguous figures to add to the area of the overall figure, there must be no overlapping. That means that the common border, a shared line, must have no area. But a line is made up of at least two points. And if points have a minimum extension, as would be the case under atomism, then the line they make up must also have a minimum area. Area, after all, is the square of extension. The area of a line cannot be less than the sum of the areas of its points. For the area of a line to actually be zero, the width of the line must be zero. But the width of a line is just its extension in another direction. For the width to be zero, extension must be zero, and that violates the presumption that there is a minimum extension.

Another way to look at this is by making an analogy between extended figures with area and extended figures with length. When a line segment is divided into shorter and shorter segments, the limiting case is a single point. If Infinite divisibility is presumed, then the limiting case has no extension; but if atomism is presumed, then the limiting case is reached after a finite number of divisions and has a minimum extension. Analogously, when an extended figure with area is divided into narrower and narrower widths, the limiting case is a single line. And this line cannot be "narrower" than the minimum atomic extension. In order for the limiting width to be zero, the division process must continue infinitely. Since a line is the limiting case of a plane figure which is being divided, a zero width is obtained only by presuming that this figure can be divided to infinity -- by presuming infinite divisibility. In other words, a line whose width (and area) is zero is obtained only by presuming infinite divisibility. As a consequence, proofs of the Pythagorean theorem that do not take into consideration the area of shared points and lines cannot be valid under atomism.

In the atomic case, a line has a small but finite width, and hence a determinate area. This means that adjacent plane figures, figures which share a common boundary line, have overlapping areas, and the above proof of the Pythagorean theorem is therefore invalid in the atomic case; it does not account for these overlapping areas. The Pythagorean theorem is used to prove the incommensurability of the diagonal with the sides of a triangle. Without the Pythagorean theorem, this incommensurability cannot be shown.

In atomic metrics, points have a minimum extension, and consequently, have a minimum length, area, and volume. If a point is extended in one direction, it must also be extended in all other directions. If it weren't, the proposition that there is a minimum extension would be contradicted. Lines, which are made up of points, must also have a minimum length, area, and volume.

As noted in the last chapter, the length of two line segments, say of length A and B, which touch at a point is the sum of the individual lengths less the extension of the common point: A + B - 1. The same principle holds for area and adjacent plane figures with a line of intersection as holds for length and lines with a point of intersection. Two plane figures which share a common boundary line have overlapping areas. The sum of the areas of two plane figures is more than the area they jointly cover by exactly the area of the line they share. The true (atomic) area of the combined figure is obtained by adding the areas of the contiguous figures and subtracting the area of their common border line.

Lines drawn in the atomic plane have a minimum area of two atomic units, since it takes two points to determine a line, but may have an area from two up to the "length" of the line; skew lines which pass "between" points are less "dense" than lines which hit the points exactly.

In general, the calculation of the area of a plane figure is not a simple matter. Moreover, determining how many points are shared by adjacent figures is no simple matter. Standard (continuous metric) mensuration formulas are based on lines having no area. Blithely using these standard continuous metric mensuration formulas when "refuting" "atomic" distances presumes infinite divisibility --

begs the question -- or worse yet, introduces an implicit contradiction.

The length of two lines which meet at a point is:

$$A + B - 1$$

The area of two figures which share line C is:

$$D + E - C$$

For a square with a side whose length is divided into A and B the area is:

$$(A + B - 1)^2 = A^2 + B^2 + 2AB - 2A - 2B + 1$$

Computing the Area of Atomic Triangles

The area of a right triangle with sides B and H is computed by the formula:

$$A = \sum_{I=1}^{B} \mathrm{INT}((I-1)(H-1)/(B-1)+1)$$

This formula does not readily render itself into a simple relation using B and H. We can transform this formula into one which can be compared with the familiar formula in continuous metrics for the area of a triangle. For the purpose of this demonstration we will not need to make the transformation into a general formula that works for all atomic triangles. We will be showing that one particular special case of atomic triangles disproves the Pythagorean theorem, so we will only need an area formula for that special case. If the Pythagorean formula is to hold, it must hold in the special case we will be considering; if it does not hold in that special case, then it cannot hold in general. By limiting our examination to this special case, our task is greatly simplified.

The length of the sides of an atomic triangle is an integer. As such, the length is either even or odd; there are just 4 possible combinations. Let the base of the triangle be B and the height of the triangle be H. We will look at the special case when B is even. This will allow us to divide it by 2 and still have an integer. The advantage to doing this will become apparent.

Since B is even, there are an even number of terms in the sum, and half an even number is also an integer. We can divide the series into two halves and then add the first and last term, the second and the next to the last, etc., ending with adding the two middle terms. To convert these terms for a sequence of integers we would substitute an expression which evaluates to the number of terms when an indexing variable is set to 1. Such a conversion is accomplished by substituting B+1-I for I. This allows us to sum to B/2.

$$\sum_{I=1}^{B} \text{INT}((I-1)(H-1)/(B-1)+1) \text{ becomes:}$$

$$\sum_{I=1}^{B/2} \text{INT}((I-1)(H-1)/(B-1)+1)+\text{INT}((B+1-I-1)(H-1)/(B-1)+1)$$

We want to get both INT terms of the expression into a form with as much similarity as possible; we want to combine like terms and remove quantities from under the INT function. As I perform a little algebra, I **bold** items of interest to make following the steps easier.

INT((I-1)(H-1)/(B-1)+1)+INT((B+1-I-1)(H-1)/(**B-1**)+1)

By rearranging we get a '(B-1)' which we can divide out.

= INT((I-1)(H-1)/(B-1)+1)+INT((<u>B-1</u>-I+1)(H-1)/(B-1)+1)

Multiply through by '1/(B-1)'.

= INT((I-1)(H-1)/(B-1)+1)+
INT((B-1)**(H-1)/(B-1)**+(-I+1)(H-1)/(B-1)+1)

Cancel out (B-1)/(B-1).

= INT((I-1)(H-1)/(B-1)+1)+INT(**H-1**+(-I+1)(H-1)/(B-1)+1)

Factor out a '-1'.

= INT((I-1)(H-1)/(B-1)+1)+INT(H-1-(I-1)(H-1)/(B-1)+1)

Integer values may be removed from the INT function, so we do so.

= INT((I-1)(H-1)/(B-1))+1+H-1+INT(-(I-1)(H-1)/(B-1))+1

Rearrange terms.

= INT((I-1)(H-1)/(B-1))+INT(-(I-1)(H-1)/(B-1))+1+H-1+1

Simplify.

= INT((I-1)(H-1)/(B-1))+INT(-(I-1)(H-1)/(B-1))+H+1

We have reduced the expression to one which is the sum of the integer part of a number plus the integer part of its negation plus an integer -- INT(Z)+INT(-Z)+N.

```
Note that INT(X.0) = X, INT(-X.0)=-X
      and INT(X.Y) = X; INT(-X.Y)=-X-1;
      so, INT(X.0)+INT(-X.0)=0
      and INT(X.Y)+INT(-X.Y)=-1.
```

The INT parts of the above expression is just 0 or -1, depending upon whether the expression (I-1)(H-1)/(B-1) evaluates to an integer or a number with a fractional part.

We are summing

INT((I-1)(H-1)/(B-1))+INT(-(I-1)(H-1)/(B-1))+H+1

from I=1 to B/2. So that we may compare our result with the familiar continuous formula for the area of a triangle, let us include the 1 with the INT expression and leave the H separate.

The INT parts of the expression evaluated to 0 or -1, so adding the 1 yields an expression which evaluates to 1 or 0. This leaves the H. Summing from I=1 to B/2 gives BH/2 plus the sum of the revised INT expression, which can be interpreted as just the amount by which the area of an atomic triangle is larger than a corresponding continuous triangle. We get:

$$BH/2 + \sum_{I=1}^{B/2} \mathrm{INT}((I-1)(H-1)/(B-1))+\mathrm{INT}(-(I-1)(H-1)/(B-1))+1$$

Without a loss of generality we can assume that $H \leq B$. By this hypothesis, H/B is either 1 or something less than 1. Since the final formula must hold for all special cases, it must also hold if H=B. To simplify our demonstration we let H=B.

If H=B then (H-1)/(B-1) is 1.[4] With this special case we will be able to completely eliminate the INT function from the expression.

```
  INT((I-1)(H-1)/(B-1))+INT(-(I-1)(H-1)/(B-1))+H+1
= INT(I-1)+INT(-(I-1))+H+1
= I-1-(I-1)+H+1
= H+1
```

Pictorially we can show what we are doing; we are "chopping off" part of a triangle, rotating it, and fitting the parts together to make a rectangle as in figure 27.

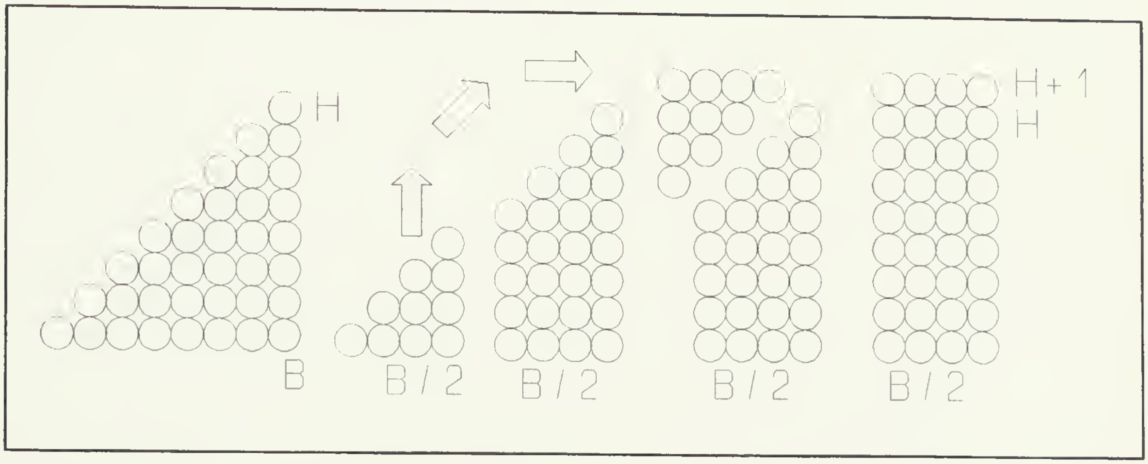

Figure 27. Pictorial representation of area computation for an atomic triangle with an even base.

$$\sum_{I=1}^{B/2} H+1 = (H+1)B/2$$

The area is BH/2 + B/2 (or BH/2 + H/2, since B=H).

Having worked out a formula for the area of certain atomic triangles, let us consider the proof of the Pythagorean theorem for the special case when B=H and both are even.

The Pythagorean Formula in Atomic Metrics

Let us now examine the same diagram which purports to demonstrate the proof of the Pythagorean theorem, by using 6 overlapping plane figures, using an atomic metric. The area of the outer square is $(A+B-1)^2 = A^2 + B^2 + 2AB -2A -2B +1$. The area of a triangle is $AB/2 + B/2$.

We will assume the area of the inner square is C^2. This assumption is not generally valid because atomic squares not aligned with the axes do not generally have square areas, as Figure 28 demonstrates.

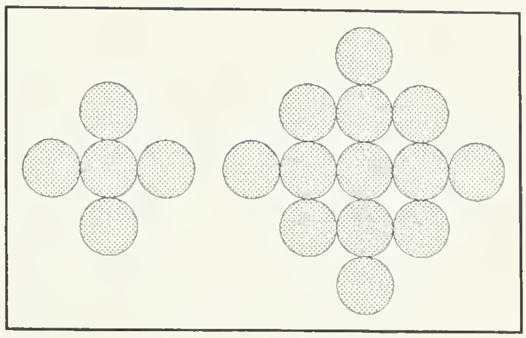

Figure 28. Squares in the atomic plane without square areas.

There are some diagonal squares with square areas though. The illustration in Figure 33 below has an offset square with an area of 16, and the inner square in Figure 31 below has an area of 25.

The number of points on the diagonal of a triangle which are in common with a side of the inner square is B. This is true even of triangles with odd lengths. Figure 29

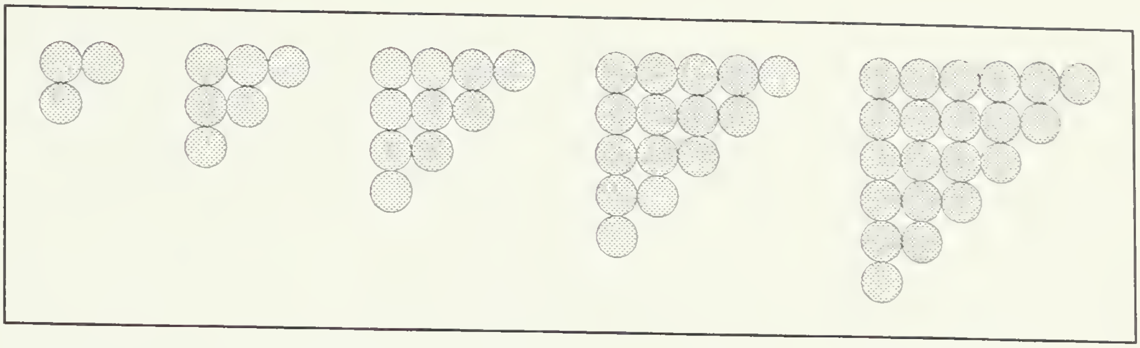

Figure 29. The area of the diagonal in atomic isosceles triangles.

illustrates this.

The triangles overlap at the end points, so four points would be counted twice and must be subtracted. Figure 30 illustrates this. Adding the four triangle areas and subtracting the area of the overlapping points gives 4•(A•B/2 + B/2) - 4.

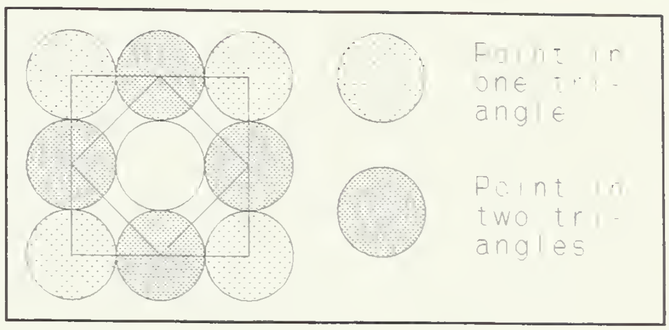

Figure 30. Intersection points of 4 triangles.

The perimeter of the inner square consists of 4 lines, each of which overlaps with a triangle and has B points. But these lines intersect with each other at the vertices of the square, so the combined area of these 4 lines themselves is 4B - 4. We already have 4(AB/2 +B/2) - 4 as the area of the four outer triangles, so we can add the area of the inner square, C^2, and subtract the area of the common boundary $(4B - 4)$, to get the combined total area. This gives us:

$$4(AB/2+B/2) - 4 + C^2 - (4B - 4)$$

Setting these two areas equal and performing algebra,

$$\begin{aligned}
\underline{\text{Outer square area}} &= \underline{\text{Four triangles}} + \underline{\text{Inner square}} \\
(A + B - 1)^2 &= 4(AB/2+B/2) - 4 + C^2 - (4B - 4) \\
A^2 + B^2 + 2AB - 2A - 2B + 1 &= 2AB + 2B - 4 + C^2 - 4B + 4 \\
A^2 + B^2 - 2A - 2B + 1 &= -2B + C^2 \\
A^2 + B^2 - 2A &= -1 + C^2 \\
A^2 + B^2 &= +2A - 1 + C^2 \\
A^2 + B^2 = C^2 + 2A - 1 &
\end{aligned}$$

Since A=B this reduces to

$$\begin{aligned}
2A^2 &= C^2 + 2A - 1 \\
2A^2 - 2A + 1 &= C^2
\end{aligned}$$

The infinitely divisible Pythagorean equivalent would be $2A^2 = C^2$. Figure 31 is an illustration with $A = B = 4$.

Since $2A^2 - 2A + 1 = C^2$ is satisfied for $A=4$ and $C=5$, we can interpret the fact that the diagonal line has only 4 points while $C = 5$ as meaning that the 4 points are "sparsely" distributed along the line that, were it aligned with the axis, would be "densely" populated with 5 points. Figure 32 shows a 5X5 square rotated 45° and superimposed upon a diagonal square with an area of 25 points. As you can see, they take up the same space as well as have the same

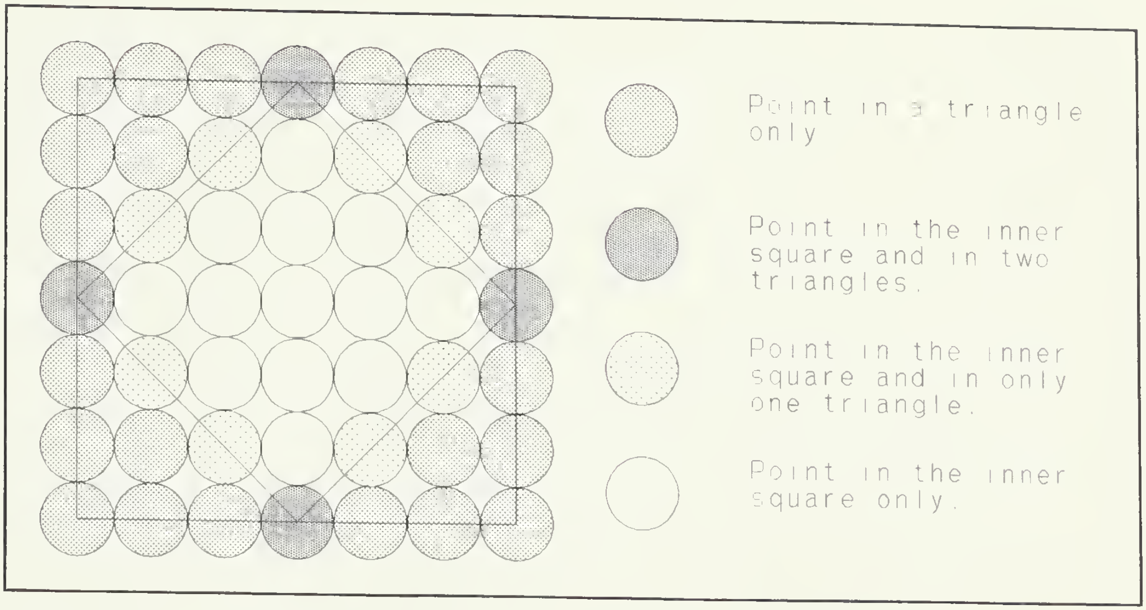

Figure 31. Illustrating the equivalent of the Pythagorean proof figure with atomic figures.

area.

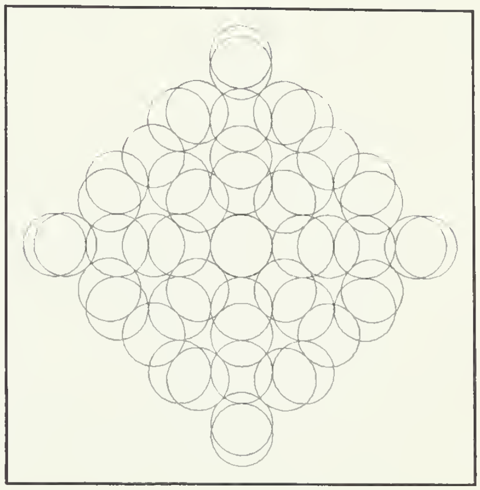

Figure 32. A 5x5 square rotated 45° and superimposed on a diagonal square with an area of 25 atomic units.

Let us look at atomic Pythagorean theorem demonstration for the simple case of A=3, B=4, and C=4[5] -- as shown in figure 33. The length of the outer square side is A + B - 1, which is 3 + 4 - 1 = 6. The area of the outer square is 36. The area of a triangle with sides 3 and 4 is 7. The four triangles touch another at four vertices, so the combined area of the four triangles is 4 times 7 less 4 or 24. The area of the inner square is 16[6]. But the outer triangles and the inner square share a common border which is only 4 points. Adding these areas and subtracting the area of the common border gives 24 + 16 - 4 = 36.

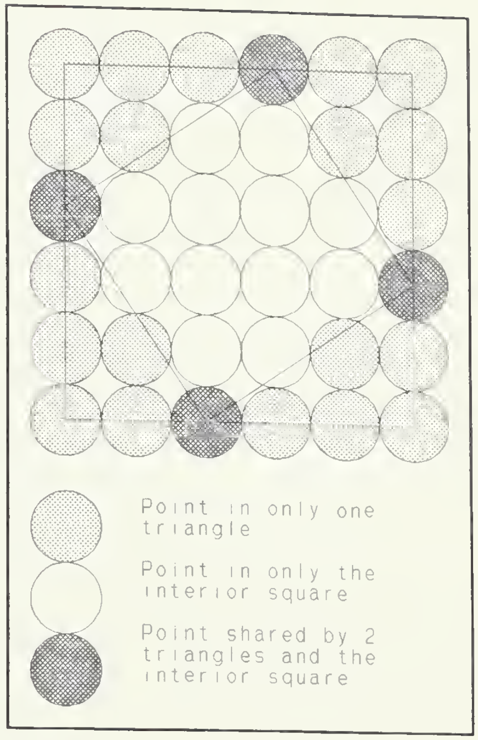

Figure 33. Atomic Pythagorean theorem demonstration for the case of A=3, B=4, and C=4.

The Pythagorean formula fails for even this special case. It is unnecessary to go through the even more complicated examples for the other cases, since the damage could not be undone. The necessary condition for the Pythagorean theorem to hold is that the intersection between two lines be of zero area. But, for that to be true, points must have zero extension; for points to have zero extension, infinite divisibility must hold.

Loomis collects and presents 370 "proofs" for the Pythagorean theorem.[7] These proofs all involve either the addition of lengths, or ratios of lengths. None of the proofs which involve the addition of lengths takes into consideration the area of the shared points. And none of the proofs which depend upon ratios makes any provision for integer results. Both conditions fail to yield the Pythagorean formula when atomic extension is considered.

Since the Pythagorean theorem does not hold in the atomic plane, it cannot be appealed to in any argument purporting to disprove atomism. Doing so begs the question.

Some Final Remarks

Is there a limit to dividing? If the object is matter, the answer is a qualified yes (according to the standard model of physics). If the object is numbers, the answer is no. But when the object is space, the answer is less clear. Epistemological concerns point out that the structure of our perceptual processes bias the answer toward yes. If we generalize away from the bias of our own perceptual process and attempt to reason toward some general way of conceptualizing space, we come to the uncertainty of what we mean by "how much" or "quantity". Typically we have determined quantity by counting or measuring. And therein lies the difficulty. We count things that are one. We measure things that are divisible. We can't answer the question of the quantity of space without choosing one of the ways we use to determine quantity. But so choosing begs the question. It now seems that thing are atoms because we count them and that things are divisible because we measure them.

Is there some a priori way to rule one way out as somehow not primary? That is, can one of the general views, atomism or divisionism, be ruled out by showing it to be flawed in some way? Over the millennia most philosophers have thought so. The pendulum has swung back and forth many times. And each swing usually followed some development in our way of viewing the world.

Prior to its expression atomism held sway in the form of Melanesian Monism. Everything was one. But monism implied that motion was impossible. So monism gave way to pluralism. But pluralism suffered from infinite regress. In a resurgence of monism, extension was claimed to be infinite. But the problem of motion remained, so space was invented. Now the problem of infinite divisibility can be applied to pure extension (without body). But each side continues to find flaws and question-begging in the other side’s arguments. At the same time we find a proliferation of mathematical tools on both sides. Measuring develops right along side counting.

Infinite sets correspond to many; singleton sets correspond to one. Either thesis (one or many) generates its antithesis (many or one), but both are required to synthesize a solution. In physics relativity theory shows the identity of matter and energy, and De Broglie developed an equation relating the wavelength of a particle to its momentum.

Modern developments in the philosophy of science show that any description we make of reality is at best a not-yet-disconfirmed model, and while we can continually improve the model by testing and revising it, there is no a priori way to know when the final model is achieved. Moreover, we know that our present model of physics is not the final model because it does not include an account of gravity.

We also now have ample evidence that both points of view (atomism and divisionism) can reliably be used in various circumstances. Although there is as yet very little general familiarity with discrete metrics, modern computer display screens are becoming more and more common, and users who deal with computer graphics are becoming much more accustomed to the discrete metrics involved.

For some things, we use the perspective of infinite divisibility; for others, we use the perspective of atomism. Although incompatible with each other, these perspectives appear to be internally self-consistent. I make an analogy using model theory. Each point of view is a language capable of describing "stuff" in a certain way. And either point of view may be taken. But as modern philosophy of science has shown, we have no access to the "stuff" except through the language we choose.

It has been said by that great philosopher Anonymous:

> There are two kinds of people in the world -- those who divide the world up into two kinds of people and those who do not.

That jest illustrates the tension that pervades our thinking. The two kinds illustrates counting and atomism while the opposition illustrates measuring and divisionism. We count and we measure, and the distinction between these pervades even our basic physical theory. Wave mechanics is contrasted with Quantum mechanics. Interference patterns in light are explained by choosing the divisionist perspective. The photo-electric effect is explained by choosing the atomist perspective. In our physical theory there is an equation to relate the two views. In mathematics we can use either to generate the other.

We may, in fact, reject the need to choose between these views, choosing each in its turn according to our needs and its efficacy in the use to which we intend to put it. Heraclitus may have said it first.

> From out of all the many particulars comes oneness, and out of oneness comes all the many particulars.[8]

Notes and References

1. Antonie Arnauld, The Art of Thinking: Port-Royal Logic, Translated by James Dickoff and Patricia James (New York: Bobbs-Merrill, 1964), p. 299.

2. Loomis, Elisha Scott, The Pythagorean Proposition, (Washington D. C.: National Council of Teachers of Mathematics, 1968), p. 224.

3. Loomis, p. 197. The proof given using figure 269 most closely conforms to the present demonstration.

4. Since B is the base of a triangle and H is its height, neither can be 1, and we needn't worry about division by zero.

5. Although C would be equal to 5 for continuous metric spaces, it is not 5 in the atomic case; C is 4 for the atomic case. Because the length of the side of the outer square formed by "adding" the lengths of the sides of the triangles is "shorter" (by 1) than it would be in continuous metrics, the size of the atomic inscribed square is smaller than its corresponding continuous analogue.

6. See Note 5.

7. Loomis

8. Philip Wheelwright, Heraclitus, (Princeton: Princeton University Press, 1959; reprint, New York: Atheneum, March 1971), p. 90.

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